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Chapman–Enskog solutions to arbitrary order in Sonine polynomials V: Summational expressions for the viscosity-related bracket integrals

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ABSTRACT

The Chapman–Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for both simple gases and gas mixtures. These coefficients include the viscosity, the thermal conductivity, and the diffusion coefficient. In a preceding paper on simple gases (I), we have shown that the use of higher-order Sonine polynomial expansions enables one to obtain results of arbitrary precision that are free of numerical error. In two subsequent papers (II–III), we extended our original simple gas work to encompass binary gas mixture computations of the viscosity, thermal conductivity, diffusion, and thermal diffusion coefficients to high-order. In a fourth paper (IV) we derived general summational representations for the diffusion- and thermal conductivity-related bracket integrals and provided compact, explicit expressions for all of these bracket integrals needed to compute the diffusion- and thermal conductivity-related transport coefficients up to order 5 in the Sonine polynomial expansions used. In all of this previous work we retained the full dependence of our solutions on the molecular masses, the molecular sizes, the mole fractions, and the intermolecular potential model via the omega integrals up to the final point of solution via matrix inversion. The elements of the matrices to be inverted are, in each case, determined by appropriate combinations of bracket integrals which contain, in general form, all of the various dependencies. Since accurate expressions for the needed bracket integrals have not previously been available in the literature beyond orders 2 or 3, and since such expressions are necessary for any extensive program of computations of the transport coefficients involving Sonine polynomial expansions to higher orders, we have investigated alternative methods of constructing appropriately general bracket integral expressions that do not rely on the term-by-term, expansion and pattern matching techniques that we developed for our previous work. It is our purpose in this paper to report the results of our efforts to obtain useful, alternative, general expressions for the bracket integrals associated with the viscosity-related Chapman–Enskog solutions for gas mixtures. Specifically, we have obtained such expressions in summational form that are conducive to use in high-order viscosity coefficient computations for arbitrary gas mixtures and have computed and reported explicit expressions for all of the orders up to 5.

1. Introduction

The Chapman–Enskog solutions of the Boltzmann equations provide a basis for the computation of important transport coefficients for both simple gases and gas mixtures [1–15]. The use of Sonine polynomial expansions for the Chapman–Enskog solutions was first suggested by Burnett [16] and has become the general method for obtaining the transport coefficients due to the relatively rapid convergence of this series [1–8,16]. While it has been found that relatively, low-order expansions (of order 4) can provide reasonable accuracy in computations of the transport coefficients (to about 1 part in 1000), most existing computer codes do not use these solutions beyond order 2 or order 3 as the relevant expressions rapidly become increasingly complex and have not been available as general, explicit expressions in terms of arbitrary potential model (via the omega integrals) in the past. Recently, our investigations of simple gases and gas mixtures [17–21] have allowed us to pursue Chapman–Enskog solutions to relatively high-orders computationally using *Mathematica*® and, thus, accurate, completely general, expressions have been obtained and used by us up to order 60 for the viscosity-related bracket integrals and up to order 70 for the diffusion- and thermal conductivity-related bracket integrals. We note that our initial work focused on the generation of the necessary bracket integral expressions via

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a method suggested in Chapman and Cowling [1] that is best described as a term-by-term, expansion and pattern matching technique. Since this method is relatively expensive in a computational sense due to the size of the higher-order bracket integral expressions, and since the size and complexity of the general expressions for the bracket integrals for gas mixtures makes them impractical to report explicitly in the open literature beyond the lowest orders even when organized into compact form, it is clearly desireable to have an alternative means of generation for the needed general bracket integral expressions that is both more compact and less cumbersome to implement as increasingly higher orders of approximation are employed in computations of the transport coefficients. Previously, we have reported the results of our efforts to develop a set of completely general expressions for the bracket integrals necessary to obtain Chapman–Enskog diffusion and thermal conductivity solutions up to any arbitrary order of expansion [21]. These results were presented as summational representations that might be implemented efficiently in a variety of different computational environments. Thus, in this paper we report on the results of our efforts to develop an additional set of completely general expressions for the bracket integrals necessary to obtain Chapman–Enskog viscosity solutions up to any arbitrary order which we have, again, been able to present as compact, summational representations that are straightforward to implement efficiently in a variety of different computational environments. In the following sections we describe the basic relationships relevant to the Chapman–Enskog solution for viscosity, the role of the bracket integrals in these solutions, the details of our derivation of alternative, summational, expressions for the viscosity-related bracket integrals, and explicit, precomputed expressions for the bracket integrals up to order 5 for use in existing computer codes where such are needed but have not previously been available beyond order 2 or order 3.

2. The basic relationships

Following the work and notations of Chapman and Cowling [1], as used in our previous work [18], we note that for binary gas mixtures the viscosity may be expressed to some order of approximation, m, in terms of Sonine polynomial expansions as:

$$[\mu]_m = p\left(x_1 b_1^{(m)} + x_2 b_{-1}^{(m)}\right), \tag{1}$$

where $x_1 = n_1/n$ and $x_2 = n_2/n$ are the component mixture fractions, n_1 and n_2 are the component number densities with $n = n_1 + n_2$ being the total number density of the mixture, m_1 and m_2 are the component molecular masses with $m_0 = m_1 + m_2$, p is the hydrostatic pressure, k is Boltzmann's constant, T is the temperature, $M_1 = m_1/m_0$, $M_2 = m_2/m_0$, and in which the quantities $b_{-1}^{(m)}$ and $b_1^{(m)}$ are expansion coefficients determined by solving the following system of algebraic equations:

$$\sum_{\substack{p=-m \\ p\neq 0}}^{+m} b_p b_{pq} = \beta_q \quad (q \neq 0), \tag{2}$$

in which:

$$\beta_1 = -\frac{5}{2}\frac{n_1}{n^2}, \quad \beta_{-1} = -\frac{5}{2}\frac{n_2}{n^2}, \quad \beta_q = 0\,(q \neq \pm 1). \tag{3}$$

We note that we have dropped the superscript (m) on the expansion coefficients $b_p^{(m)}$ as is done in Chapman and Cowling. In matrix notation, this system of equations may be written as:

$$\mathbf{Bb} = \boldsymbol{\beta}, \tag{4}$$

where:

$$\mathbf{B} = \begin{bmatrix} b_{-m-m} & \cdots & b_{-m-1} & b_{-m1} & \cdots & b_{-mm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{-1-m} & \cdots & b_{-1-1} & b_{-11} & \cdots & b_{-1m} \\ b_{1-m} & \cdots & b_{1-1} & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{m-m} & \cdots & b_{m-1} & b_{m1} & \cdots & b_{mm} \end{bmatrix}, \tag{5}$$

$$\mathbf{b} = \begin{bmatrix} b_{-m} \\ \vdots \\ b_{-1} \\ b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad \text{and} \quad \boldsymbol{\beta} = \begin{bmatrix} 0 \\ \vdots \\ \beta_{-1} \\ \beta_1 \\ \vdots \\ 0 \end{bmatrix}, \tag{6}$$

and where, as the order of the expansion, m, increases, the matrices build outward from their centers in the manner indicated. Thus, to obtain the needed expansion coefficients, $b_{-1}^{(m)}$ and $b_1^{(m)}$, and hence the viscosity for a given order of the expansion, one need only generate the $[(2m)\times(2m)]$ matrix of Eq. (5) and invert it.

The matrix elements, b_{pq}, in Eq. (5) are constructed from combinations of bracket integrals containing the appropriate Sonine polynomials from the expansions used. Specifically, since it is straightforward to show for any (p, q) that $b_{pq} = b_{qp}$, one has that:

$$\begin{aligned} b_{pq} = b_{qp} &= x_1^2\left[S_{5/2}^{(p-1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1, S_{5/2}^{(q-1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1\right]_1 \\ &\quad + x_1x_2\left[S_{5/2}^{(p-1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1, S_{5/2}^{(q-1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1\right]_{12}, \end{aligned} \tag{7}$$

$$b_{p-q} = b_{-qp} = x_1x_2\left[S_{5/2}^{(p-1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1, S_{5/2}^{(q-1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2\right]_{12}, \tag{8}$$

$$b_{-pq} = b_{q-p} = x_1x_2\left[S_{5/2}^{(p-1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2, S_{5/2}^{(q-1)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1^{\circ}\boldsymbol{\mathscr{C}}_1\right]_{21}, \tag{9}$$

$$\begin{aligned} b_{-p-q} = b_{-q-p} &= x_2^2\left[S_{5/2}^{(p-1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2, S_{5/2}^{(q-1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2\right]_2 \\ &\quad + x_1x_2\left[S_{5/2}^{(p-1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2, S_{5/2}^{(q-1)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2^{\circ}\boldsymbol{\mathscr{C}}_2\right]_{21}, \end{aligned} \tag{10}$$

where:

$$\begin{aligned} S_m^{(n)}(x) &= \sum_{p=0}^{n}\frac{(m+n)_{(n-p)}}{(p)!(n-p)!}(-x)^p \\ &= \sum_{p=0}^{n}\frac{(m+n)!}{(p)!(n-p)!(m+p)!}(-x)^p \\ &= \sum_{p=0}^{n}\frac{\Gamma(m+n+1)}{\Gamma(p+1)\Gamma(n-p+1)\Gamma(m+p+1)}(-x)^p, \end{aligned} \tag{11}$$

(with $S_m^{(0)}(x) = 1$ and $S_m^{(1)}(x) = m + 1 - x$) are numerical multiples (un-normalized) of the Sonine polynomials originally used in the Kinetic Theory of Gases by Burnett [16], in which $\boldsymbol{\mathscr{C}}_i = (m_i/2kT)^{1/2}\mathbf{C}_i$ are dimensionless, pre-collision, peculiar molecular velocities, $\mathbf{C}_i = \mathbf{c}_i - \mathbf{c}_0$ are the dimensional, pre-collision, peculiar molecular velocities, $\mathbf{c}_i$ are the pre-collision molecular velocities, and $\mathbf{c}_0 = M_1x_1\mathbf{c}_1 + M_2x_2\mathbf{c}_2$ is the mean mass velocity of the mixture. Here, one needs to be aware that the notation $(m+n)_{(n-p)}$ used by Chapman and Cowling in Eq. (11) is not the standard Pochhammer notation employed later in this work although it is related to it. From the definitions used in the bracket integral notation, it follows that Eqs. (10) and (9) are essentially identical to Eqs. (7) and (8), respectively, with the only difference being the interchange of the subscripts 1 and 2 representing the different components of the mixture. Thus, in general, the complete Chapman–Enskog solution for viscosity for binary gas mixtures requires the evaluation of only three types of bracket integrals:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_1, \tag{12}$$

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\overset{\circ}{\boldsymbol{\mathscr{C}}}_2\right]_{12}, \tag{13}$$

and:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\overset{\circ}{\boldsymbol{\mathscr{C}}}_1\right]_{12}. \tag{14}$$

Here, however, we note an observation made in our previous work [18] that, of the three required bracket integrals shown in Eqs. (12)–(14), the application of some simple combinatorial rules allows one to generate expressions for the bracket integrals of Eqs. (12) and (13) from an appropriate expression for the bracket integral of Eq. (14) if such is known. Thus, it is the derivation of such an expression for the bracket integral of Eq. (14) which could be said to be the most important goal of this paper. However, in terms of complexity, we have found it convenient to first pursue independently the derivation of an expression for the bracket integral of Eq. (13) before attempting the corresponding derivation for the bracket integral of Eq. (14). Both of these derivations are detailed below and the similarities between them are obvious. Our use of the resulting expressions for the bracket integrals of Eqs. (13) and (14) to generate a similar expression for the bracket integral of Eq. (12) according to the appropriate combinatorial rule is a relatively minor exercise which is presented following the first two derivations. While all of the above expressions for binary gas mixtures are readily generalized for use with arbitrary mixtures by replacing the (1, 2) indexing scheme associated specifically with binary mixtures to a more general (i, j) indexing scheme, in what follows we retain the (1, 2) indexing scheme used by Chapman and Cowling as it improves (in our opinion) the clarity of the derivations. Of great importance in this work is the requirement that we have placed on our results that they continue to exhibit the full set of general dependencies of the bracket integrals on the molecular masses and the omega integrals, $\Omega_{12}^{(\ell)}(r)$, that we have retained in our previous recent work [17–21] and which is the most significant factor contributing to the utility of this recent body of work. The omega integrals are initially defined as:

$$\Omega_{12}^{(\ell)}(r) \equiv \left(\frac{kT}{2\pi m_0 M_1 M_2}\right)^{1/2} \int_0^\infty \exp\left(-g^2\right) g^{(2r+3)} \phi_{12}^{(\ell)} \mathrm{d}g, \tag{15}$$

with:

$$\phi_{12}^{(\ell)} \equiv 2\pi \int_0^\pi \left[1 - \cos^\ell(\chi)\right] b \mathrm{d}b, \tag{16}$$

and contain all of the dependencies relating to the specific intermolecular potential model that is employed. Here, χ is the angle between the pre-collision ($\mathbf{g} = \mathbf{c}_2 - \mathbf{c}_1$) and post collision ($\mathbf{g}' = \mathbf{c}_2' - \mathbf{c}_1'$) relative velocities and is a function of the impact parameter, b, and the dimensionless pre-collision, relative velocity of the two colliding molecules, $g \equiv (m_0 M_1 M_2/2kT)^{1/2}\mathbf{g}$. As a brief aside, we note here that it is often considered convenient to define the omega integrals in terms of a simple scaling factor, $\sigma_{12} = (½)(\sigma_1 + \sigma_2)$, which is, in the most general of terms, only a convenient, arbitrarily chosen length within some range where the impact parameter, b, is significant. Expressed in this manner, the omega integrals are then [1]:

$$\Omega_{12}^{(\ell)}(r) = \frac{1}{2}\sigma_{12}^2 \left(\frac{2\pi kT}{m_0 M_1 M_2}\right)^{1/2} W_{12}^{(\ell)}(r), \tag{17}$$

where:

$$W_{12}^{(\ell)}(r) \equiv 2\int_0^\infty \exp\left(-g^2\right) g^{(2r+3)} \times \int_0^\pi \left[1 - \cos^\ell(\chi)\right](b/\sigma_{12})\mathrm{d}(b/\sigma_{12})\mathrm{d}g. \tag{18}$$

Note that when only one species is present, Eqs. (17) and (18) reduce to the following simple gas expressions:

$$\Omega_1^{(\ell)}(r) = \sigma_1^2 \left(\frac{\pi kT}{m_1}\right)^{1/2} W_1^{(\ell)}(r), \tag{19}$$

with:

$$W_1^{(\ell)}(r) = 2\int_0^\infty \exp\left(-g^2\right) g^{(2r+3)} \times \int_0^\pi \left[1 - \cos^\ell(\chi)\right](b/\sigma_1)\mathrm{d}(b/\sigma_1)\mathrm{d}g. \tag{20}$$

In Eqs. (17)–(20), σ_1 is an arbitrary scale length associated with collisions between like molecules of type 1 while σ_{12} is associated with collisions between unlike molecules of types 1 and 2. These scale lengths are commonly associated with some concept of the molecular diameters depending upon the specific details of the intermolecular potential model that is employed.

3. Derivation of summational representations for the bracket integrals

We begin our derivations at the point in Chapman and Cowling [1] where the evaluation of the six integrations in the bracket integrals that are unrelated to the intermolecular interaction model being employed are first considered. Here, we note that Chapman and Cowling make use of the following relationships for the Sonine polynomials:

$$\left(\frac{\mathrm{S}}{s}\right)^{(m+1)} \exp(-x\mathrm{S}) = (1-s)^{(-m-1)} \exp\left(\frac{-xs}{1-s}\right) = \sum_{n=0}^\infty s^n S_m^{(n)}(x), \tag{21}$$

and:

$$\left(\frac{\mathrm{T}}{t}\right)^{(m+1)} \exp(-x\mathrm{T}) = (1-t)^{(-m-1)} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^\infty t^n S_m^{(n)}(x), \tag{22}$$

where $\mathrm{S} = s/(1-s)$ and $\mathrm{T} = t/(1-t)$, to express the bracket integrals in terms of the coefficients of expansions in the arbitrarily introduced variables, s and t. Thus, it is possible after following Chapman and Cowling to obtain the following expressions for the bracket integrals of Eqs. (13) and (14), respectively:

$$\left[S^{(p)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(q)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12} = \text{coeff}[s^p t^q] \text{ in } \left(\frac{ST}{st}\right)^{7/2}\pi^{-3}\int\int\int[L_{12}(0)-L_{12}(\chi)]gb\,db\,d\varepsilon\,d\mathscr{g}, \quad (23)$$

and:

$$\left[S^{(p)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(q)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} = \text{coeff}[s^p t^q] \text{ in } \left(\frac{ST}{st}\right)^{7/2}\pi^{-3}\int\int\int[L_1(0)-L_1(\chi)]gb\,db\,d\varepsilon\,d\mathscr{g}. \quad (24)$$

Note that the retention of a single g in Eqs. (23) and (24) (as opposed to $\mathscr{g}$) is not a typographical error but, rather, is the exact notation used by Chapman and Cowling. Then, one can express the χ-dependent portions of the RHS bracketed integrals of Eqs. (23) and (24) as:

$$\begin{aligned}\frac{3}{2}\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi) &= (M_1M_2)\exp\left(-\mathscr{g}^2\right) \\ &\times\sum_r\sum_n\{2M_1M_2st[1-\cos(\chi)]\}^r\left(\frac{\mathscr{g}^{2r}}{r!}\right) \\ &\times(M_2s+M_1t)^n\Big\{[(n+1)(n+2)]S^{(n+2)}_{r+1/2}\left(\mathscr{g}^2\right) \\ &+[2(n+1)[1-\cos(\chi)]]\mathscr{g}^2S^{(n+1)}_{r+3/2}\left(\mathscr{g}^2\right) \\ &+\left[[1-\cos(\chi)]^2-\frac{1}{2}\sin^2(\chi)\right]\mathscr{g}^4S^{(n)}_{r+5/2}\left(\mathscr{g}^2\right)\Big\}, \end{aligned} \quad (25)$$

and:

$$\begin{aligned}\frac{3}{2}\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) &= \exp\left(-\mathscr{g}^2\right) \\ &\times\sum_r\sum_n\left\{st[M_1^2+M_2^2+2M_1M_2\cos(\chi)]\right\}^r\left(\frac{\mathscr{g}^{2r}}{r!}\right) \\ &\times\{M_2(s+t)-(M_2-M_1)st\}^n\Big\{\left[M_1^2(n+1)(n+2)\right]S^{(n+2)}_{r+1/2}\left(\mathscr{g}^2\right) \\ &+[2(n+1)M_1[M_1+M_2\cos(\chi)]]\mathscr{g}^2S^{(n+1)}_{r+3/2}\left(\mathscr{g}^2\right) \\ &+\left[[M_1+M_2\cos(\chi)]^2-\frac{1}{2}M_2^2\sin^2(\chi)\right]\mathscr{g}^4S^{(n)}_{r+5/2}\left(\mathscr{g}^2\right)\Big\}, \end{aligned} \quad (26)$$

which are Eqs. (9.32, 7) and (9.4, 13), respectively, in Chapman and Cowling. In both of these cases, the coefficient of $[s^p t^q]$ yields a polynomial in powers of $\mathscr{g}^2$ and $\cos(\chi)$ that is multiplied by $\exp(-\mathscr{g}^2)$ and in which each term is some function of the molecular masses via M_1 and M_2. The χ-independent portions of the RHS bracketed integrals of Eqs. (23) and (24) are obtained by the simple expedient of setting $\chi=0$ in Eqs. (25) and (26) which then yields overall terms in the combined polynomial involving $[1-\cos^\ell(\chi)]$. Thus, after completion of the six integrations not related to the intermolecular potential model, including the integrations over ε and the directions of $\mathscr{g}$, it is possible to express the bracket integrals of Eqs. (23) and (24) as:

$$\begin{aligned}&\left[S^{(p)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(q)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12} \\ &= \frac{16}{3}\pi^{1/2}M_2^{(p+1)}M_1^{(q+1)}\int\int\exp\left(-\mathscr{g}^2\right) \\ &\times\sum_{r,\ell}B_{pqr\ell}\,\mathscr{g}^{(2r+2)}\left[1-\cos^\ell(\chi)\right]gb\,db\,d\mathscr{g} \\ &= \frac{16}{3}M_2^{(p+1)}M_1^{(q+1)}\sum_{r,\ell}B_{pqr\ell}\,\Omega^{(\ell)}_{12}(r) = \frac{16}{3}\sum_{r,\ell}B''_{pqr\ell}\,\Omega^{(\ell)}_{12}(r), \end{aligned} \quad (27)$$

and:

$$\begin{aligned}&\left[S^{(p)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(q)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} = \frac{16}{3}\pi^{1/2}\int\int\exp\left(-\mathscr{g}^2\right) \\ &\times\sum_{r,\ell}B'_{pqr\ell}\,\mathscr{g}^{(2r+2)}\left[1-\cos^\ell(\chi)\right]gb\,db\,d\mathscr{g} \\ &= \frac{16}{3}\sum_{r,\ell}B'_{pqr\ell}\,\Omega^{(\ell)}_{12}(r), \end{aligned} \quad (28)$$

where the omega integrals have been defined in Eqs. (17) and (18). Here, we note that our initial work followed the prescription implied by Chapman and Cowling [1] for determination of the coefficients, $B_{pqr\ell}$ and $B'_{pqr\ell}$. According to Chapman and Cowling:

> "Explicit expressions for $[B_{pqr\ell}$ and $B'_{pqr\ell}]$ can be obtained from [Eqs. (25) and (26)] using [Eq. (11)] for $S^{(n)}_m(x)$. In view of the complication of these expressions it is, however, better in practice to calculate any desired values of $[B_{pqr\ell}$ and $B'_{pqr\ell}]$ directly from [Eqs. (25) and (26)]."

The prescription implied by this statement is that one should expand Eqs. (25) and (26) directly in powers of s and t using the binomial theorem, collect terms containing identical powers of $[s^p t^q]$ to identify the omega integrals that are present, and then consolidate the coefficients of each to create the needed expressions for each of the (p, q) bracket integrals and their associated b_{pq} matrix elements. In general, this process works fine for lower-order expansions; particularly where one is required to do the algebra by hand, and is readily accomplished to much higher orders of expansion by using *Mathematica*® to do the necessary algebra and pattern matching. However, at sufficiently high an order, the computational overhead associated with performing these operations on extremely large and complex expressions causes the process to become very inefficient in terms of the time required to determine the matrix elements. Thus, we return to the above quote by Chapman and Cowling and consider the alternative prescription that they have indicated which would be expected to yield general expressions for the bracket integrals much more conducive to efficient computations; particularly in computational environments employing more traditional languages and programming structures (such as FORTRAN, C++, etc.).

We return now to Eqs. (27) and (28). As we have pointed out, it is technically only necessary to actually derive an alternative expression for the coefficients $B'_{pqr\ell}$ in Eq. (28) as the coefficients $B_{pqr\ell}$ in Eq. (27) can then be determined from the $B'_{pqr\ell}$ expression thus derived. In practice, however, an expression for $B_{pqr\ell}$ is easier to derive due to its less complex dependence on the molecular masses. Therefore, $B_{pqr\ell}$ is addressed first followed by $B'_{pqr\ell}$. With both of these coefficients determined, the bracket integrals of Eqs. (13) and (14) are fully specified in the most general possible terms and may be combined in the appropriate manner to yield the most general possible expression for the simple gas bracket integral of Eq. (12). Then, with all three of the bracket integrals of Eqs. (12)–(14) thus specified, general expressions for the b_{pq} matrix elements may be constructed according to Eqs. (7)–(10) if one wishes. At this point, evaluation of the general matrix elements for specific values of the parameters and inversion of the coefficient matrices to obtain the expansion coefficients, $b^{(m)}_{-1}$ and $b^{(m)}_1$, is an extremely rapid process provided that values of the necessary omega integrals exist in precomputed form to the necessary degree of precision for the specific intermolecular potential model being employed.

4. Derivation of a summational representation for the L_{12} bracket integral

First, consider the bracket integral type:

$$\left[S^{(p)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(q)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}, \quad (29)$$

which we refer to here as the L_{12} bracket integral. Following Eq. (23) this may be determined by specifying the coefficient of $[s^p t^q]$ in the expansion of Eq. (25). As a first step one may rewrite Eq. (25) in the following slightly more convenient form:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-g^2\right)$$
$$\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}(M_1M_2)^i s^i t^i\frac{2^i}{(i)!}[1-\cos(\chi)]^i\left(g^2\right)^i$$
$$\times(M_2s+M_1t)^n\left\{\left(n+1\right)\left(n+2\right)S_{i+1/2}^{(n+2)}\left(g^2\right)\right.$$
$$+2(n+1)[1-\cos(\chi)]g^2S_{i+3/2}^{(n+1)}\left(g^2\right)$$
$$\left.+\left[\frac{1}{2}-2\cos(\chi)+\frac{3}{2}\cos^2(\chi)\right]g^4S_{i+5/2}^{(n)}\left(g^2\right)\right\}. \tag{30}$$

Now, one consolidates the Sonine polynomials using the definition of Eq. (11) which may be expressed as:

$$S_{z+1/2}^{(n+2)}\left(g^2\right)=\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^\eta}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{(2(z+n+3))!}{(2(z+\eta+1))!}\frac{(z+\eta+1)!}{(z+n+3)!}\frac{4^\eta}{4^{(n+2)}}, \tag{31}$$

where we note that we have used the following property of the Gamma function:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{4^n(n)!}\Gamma\left(\frac{1}{2}\right). \tag{32}$$

From Eq. (31), one may obtain:

$$S_{i+1/2}^{(n+2)}\left(g^2\right)=\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^\eta}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{(2(i+n+3))!}{(2(i+\eta+1))!}\frac{(i+\eta+1)!}{(i+n+3)!}\frac{4^\eta}{4^{(n+2)}}, \tag{33}$$

$$g^2S_{i+3/2}^{(n+1)}\left(g^2\right)=\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^\eta}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{(2(i+n+3))!}{(2(i+\eta+1))!}\frac{(i+\eta+1)!}{(i+n+3)!}\frac{4^\eta}{4^{(n+2)}}[-\eta], \tag{34}$$

and:

$$g^4S_{i+5/2}^{(n)}\left(g^2\right)=\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^\eta}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{(2(i+n+3))!}{(2(i+\eta+1))!}\frac{(i+\eta+1)!}{(i+n+3)!}\frac{4^\eta}{4^{(n+2)}}[\eta(\eta-1)]. \tag{35}$$

After this substitution, one can factor out the common terms in the three summations and combine them into a single summation as:

$$(n+1)(n+2)S_{i+1/2}^{(n+2)}\left(g^2\right)+2(n+1)[1-\cos(\chi)]g^2S_{i+3/2}^{(n+1)}\left(g^2\right)$$
$$+\left[\frac{1}{2}-2\cos(\chi)+\frac{3}{2}\cos^2(\chi)\right]g^4S_{i+5/2}^{(n)}\left(g^2\right)$$
$$=\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^\eta}{(\eta)!(n+2-\eta)!}\frac{(2(i+n+3))!}{(2(i+\eta+1))!}\frac{(i+\eta+1)!}{(i+n+3)!}$$
$$\times\frac{4^\eta}{4^{(n+2)}}\left\{\left[(n+1-\eta)(n+2-\eta)-\frac{1}{2}\eta(\eta-1)\right]\right.$$
$$\left.+[2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}. \tag{36}$$

Now, one can write:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-g^2\right)$$
$$\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}(M_1M_2)^i s^i t^i\frac{2^i}{(i)!}[1-\cos(\chi)]^i$$
$$\times(M_2s+M_1t)^n\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{4^\eta}{4^{(n+2)}}\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(n+1-\eta)(n+2-\eta)-\frac{1}{2}\eta(\eta-1)\right]\right.$$
$$\left.+[2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}, \tag{37}$$

To extract the summation over s, one first substitutes the binomial expansion:

$$(M_2s+M_1t)^n=\sum_{j=0}^{n}\binom{n}{j}M_2^js^jM_1^{(n-j)}t^{(n-j)}, \tag{38}$$

where the $\binom{n}{j}$ are the binomial coefficients such that:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-g^2\right)$$
$$\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{2^i}{(i)!}[1-\cos(\chi)]^i\sum_{j=0}^{n}\binom{n}{j}M_2^{(j+i)}s^{(j+i)}$$
$$\times M_1^{(n-j+i)}t^{(n-j+i)}\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{4^\eta}{4^{(n+2)}}\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(n+1-\eta)(n+2-\eta)-\frac{1}{2}\eta(\eta-1)\right]\right.$$
$$\left.+[2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}. \tag{39}$$

Then, shifting the j index one has that:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)$$
$$=\frac{2}{3}(M_1M_2)\exp\left(-\mathscr{g}^2\right)\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=i}^{(n+i)}s^j\frac{2^i}{(i)!}[1-\cos(\chi)]^i\binom{n}{(j-i)}M_2^jM_1^{(n-j+2i)}t^{(n-j+2i)}\sum_{\eta=0}^{(n+2)}\frac{(-1)^{\eta}\left(\mathscr{g}^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{4^{\eta}}{4^{(n+2)}}\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\left\{\left[(n+1-\eta)(n+2-\eta)-\frac{1}{2}\eta(\eta-1)\right]+[2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}. \quad (40)$$

Now, from Figs. 1 and 2, it can be seen that:

$$\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=i}^{(n+i)}=\sum_{j=0}^{\infty}\sum_{i=0}^{j}\sum_{n=(j-i)}^{\infty}. \quad (41)$$

Thus:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-\mathscr{g}^2\right)$$
$$\times\sum_{j=0}^{\infty}s^j\sum_{i=0}^{j}\sum_{n=(j-i)}^{\infty}\frac{2^i}{(i)!}[1-\cos(\chi)]^i\binom{n}{(j-i)}M_2^j$$
$$\times M_1^{(n-j+2i)}t^{(n-j+2i)}\sum_{\eta=0}^{(n+2)}\frac{(-1)^{\eta}\left(\mathscr{g}^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{4^{\eta}}{4^{(n+2)}}\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(n+1-\eta)(n+2-\eta)-\frac{1}{2}\eta(\eta-1)\right]\right.$$
$$\left.+[2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}, \quad (42)$$

and one need only let $j\to p$ to obtain the coefficient of s^p, i.e.:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-\mathscr{g}^2\right)$$
$$\times\sum_{p=0}^{\infty}s^p\sum_{i=0}^{p}\sum_{n=(p-i)}^{\infty}t^{(n-p+2i)}\frac{2^i}{(i)!}[1-\cos(\chi)]^i$$
$$\times\binom{n}{(p-i)}M_2^pM_1^{(n-p+2i)}\sum_{\eta=0}^{(n+2)}\frac{(-1)^{\eta}\left(\mathscr{g}^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}$$
$$\times\frac{4^{\eta}}{4^{(n+2)}}\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(n+1-\eta)(n+2-\eta)-\frac{1}{2}\eta(\eta-1)\right]\right.$$
$$\left.+[2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}. \quad (43)$$

Next, to extract the summation over t, one shifts the n index such that:

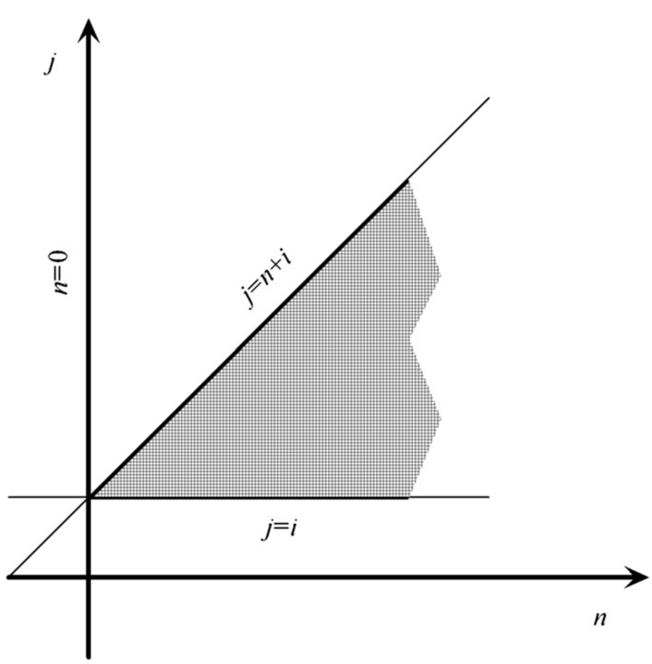

Fig. 1. The geometry of the summational transformation: $\sum_{n=0}^{\infty}\sum_{j=i}^{(n+i)}=\sum_{j=i}^{\infty}\sum_{n=(j-i)}^{\infty}$.

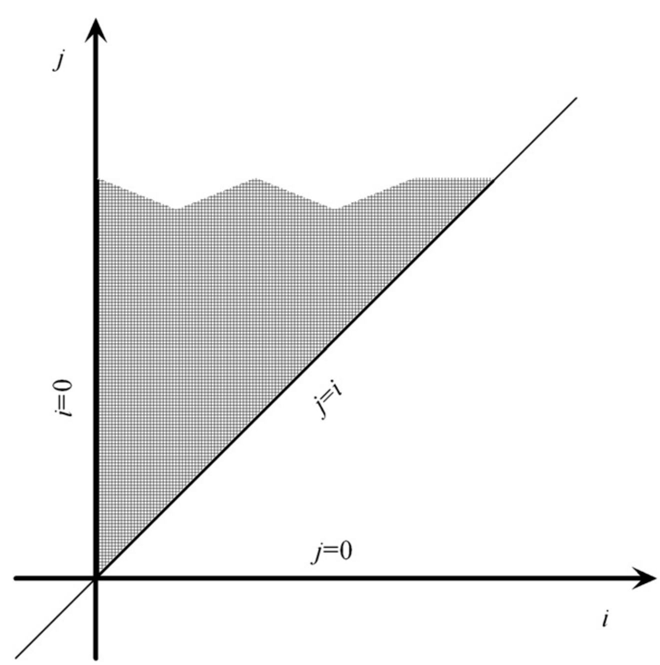

Fig. 2. The geometry of the summational transformation: $\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}=\sum_{j=0}^{\infty}\sum_{i=0}^{j}$.

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-g^2\right)$$
$$\times\sum_{p=0}^{\infty}s^p\sum_{i=0}^{p}\sum_{n=i}^{\infty}t^n[1-\cos(\chi)]^i\binom{(p+n-2i)}{(p-i)}\frac{2^i}{(i)!}M_2^pM_1^n\sum_{\eta=0}^{(p+n+2-2i)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(p+n+2-2i-\eta)!}\frac{4^\eta}{4^{(p+n+2-2i)}}\frac{(2(p+n+3-i))!}{(p+n+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+n+1-2i-\eta)(p+n+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]+[2\eta(p+n+2-2i-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}. \quad (44)$$

Here, from Fig. 3, one has that:

$$\sum_{i=0}^{p}\sum_{n=i}^{\infty}=\sum_{n=0}^{\infty}\sum_{i=0}^{\min[p,n]}. \quad (45)$$

Thus:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-g^2\right)$$
$$\times\sum_{p=0}^{\infty}s^p\sum_{n=0}^{\infty}t^n\sum_{i=0}^{\min[p,n]}[1-\cos(\chi)]^i\binom{(p+n-2i)}{(p-i)}$$
$$\times\frac{2^i}{(i)!}M_2^pM_1^n\sum_{\eta=0}^{(p+n+2-2i)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(p+n+2-2i-\eta)!}$$
$$\times\frac{4^\eta}{4^{(p+n+2-2i)}}\frac{(2(p+n+3-i))!}{(p+n+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+n+1-2i-\eta)(p+n+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]+[2\eta(p+n+2-2i-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}, \quad (46)$$

and one need only let $n\to q$ to obtain the coefficient of t^q, i.e.:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}(M_1M_2)\exp\left(-g^2\right)$$
$$\times\sum_{p=0}^{\infty}s^p\sum_{q=0}^{\infty}t^q\sum_{i=0}^{\min[p,q]}[1-\cos(\chi)]^i\binom{(p+q-2i)}{(p-i)}$$
$$\times\frac{2^i}{(i)!}M_2^pM_1^q\sum_{\eta=0}^{(p+q+2-2i)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta)!}$$
$$\times\frac{4^\eta}{4^{(p+q+2-2i)}}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+q+1-2i-\eta)(p+q+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]+[2\eta(p+q+2-2i-\eta)]\cos(\chi)+\left[\frac{3}{2}\eta(\eta-1)\right]\cos^2(\chi)\right\}. \quad (47)$$

To extract the $\cos(\chi)$ summation, one must first factor out powers of the $\cos(\chi)$ from the last term as a summation over Kronecker deltas in the following manner:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]$$
$$\times M_2^{(p+1)}M_1^{(q+1)}\exp\left(-g^2\right)$$
$$\times\sum_{i=0}^{\min[p,q]}[1-\cos(\chi)]^i\frac{2^i}{(i)!}\frac{(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\sum_{\eta=0}^{(p+q+2-2i)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta)!}\frac{4^\eta}{4^{(p+q+2-2i)}}$$
$$\times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\sum_{j=0}^{2}\cos^j(\chi)$$
$$\times\left\{\left[(p+q+1-2i-\eta)(p+q+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]\delta_{j,0}+[2\eta(p+q+2-2i-\eta)]\delta_{j,1}+\left[\frac{3}{2}\eta(\eta-1)\right]\delta_{j,2}\right\}. \quad (48)$$

Substitution of the binomial expansion:

$$[1-\cos(\chi)]^i=\sum_{k=0}^{i}\binom{i}{k}(-1)^k\cos^k(\chi), \quad (49)$$

then yields:

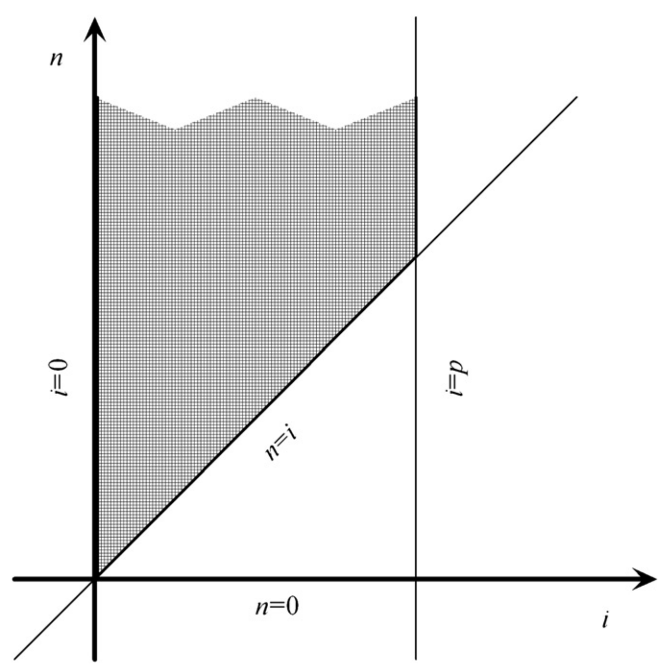

Fig. 3. The geometry of the summational transformation: $\sum_{i=0}^{p}\sum_{n=0}^{\infty}=\sum_{n=0}^{\infty}\sum_{i=0}^{\min[p,n]}$.

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)$$
$$=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathcal{g}^2\right)\sum_{i=0}^{\min[p,q]}\sum_{k=0}^{i}\sum_{j=0}^{2}\cos^{(j+k)}(\chi)\binom{i}{k}(-1)^k\frac{2^i}{(i)!}\frac{(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\sum_{\eta=0}^{(p+q+2-2i)}\frac{(-1)^{\eta}\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta)!}\frac{4^{\eta}}{4^{(p+q+2-2i)}}\frac{(2(p+n+3-i))!}{(p+n+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+q+1-2i-\eta)(p+q+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]\delta_{j,0}+[2\eta(p+q+2-2i-\eta)]\delta_{j,1}+\left[\frac{3}{2}\eta(\eta-1)\right]\delta_{j,2}\right\},\tag{50}$$

which, following a shift of the j index, becomes:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)$$
$$=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathcal{g}^2\right)\sum_{i=0}^{\min[p,q]}\sum_{k=0}^{i}\sum_{j=k}^{(k+2)}\cos^{j}(\chi)\frac{(-1)^k2^i}{(k)!(i-k)!}\frac{(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\sum_{\eta=0}^{(p+q+2-2i)}\frac{(-1)^{\eta}\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta)!}\frac{4^{\eta}}{4^{(p+q+2-2i)}}\frac{(2(p+n+3-i))!}{(p+n+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+q+1-2i-\eta)(p+q+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]\delta_{(j-k),0}+[2\eta(p+q+2-2i-\eta)]\delta_{(j-k),1}+\left[\frac{3}{2}\eta(\eta-1)\right]\delta_{(j-k),2}\right\}.\tag{51}$$

Now, from Figs. 4 and 5, one has that:

$$\sum_{i=0}^{\min[p,q]}\sum_{k=0}^{i}\sum_{j=k}^{(k+2)}=\sum_{j=0}^{(\min[p,q]+2)}\sum_{i=\max[0,(j-2)]}^{\min[p,q]}\sum_{k=\max[0,(j-2)]}^{\min[j,i]},\tag{52}$$

such that:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)$$
$$=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathcal{g}^2\right)\sum_{j=0}^{(\min[p,q]+2)}\cos^{j}(\chi)\sum_{i=\max[0,(j-2)]}^{\min[p,q]}\sum_{k=\max[0,(j-2)]}^{\min[j,i]}\frac{(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\frac{(-1)^k2^i}{(k)!(i-k)!}\sum_{\eta=0}^{(p+q+2-2i)}\frac{(-1)^{\eta}\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta)!}\frac{4^{\eta}}{4^{(p+q+2-2i)}}\frac{(2(p+n+3-i))!}{(p+n+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+q+1-2i-\eta)(p+q+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]\delta_{k,j}+[2\eta(p+q+2-2i-\eta)]\delta_{k,(j-1)}+\left[\frac{3}{2}\eta(\eta-1)\right]\delta_{k,(j-2)}\right\},\tag{53}$$

and one need only let $j\rightarrow\ell$ to obtain the coefficient of $\cos^{\ell}(\chi)$, i.e.:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_{12}(\chi)$$
$$=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathcal{g}^2\right)\sum_{\ell=0}^{(\min[p,q]+2)}\cos^{\ell}(\chi)\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}\frac{(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\frac{(-1)^k2^i}{(k)!(i-k)!}\sum_{\eta=0}^{(p+q+2-2i)}\frac{(-1)^{\eta}\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta)!}\frac{4^{\eta}}{4^{(p+q+2-2i)}}\frac{(2(p+n+3-i))!}{(p+n+3-i)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}$$
$$\times\left\{\left[(p+q+1-2i-\eta)(p+q+2-2i-\eta)-\frac{1}{2}\eta(\eta-1)\right]\delta_{k,\ell}+[2\eta(p+q+2-2i-\eta)]\delta_{k,(\ell-1)}+\left[\frac{3}{2}\eta(\eta-1)\right]\delta_{k,(\ell-2)}\right\}.\tag{54}$$

Here, note that the full integration involves the difference $[L_{12}(0)-L_{12}(\chi)]$ which yields terms containing $[1-\cos^{\ell}(\chi)]$. When $\ell=0$, this quantity is identically zero and, hence, without loss of generality, one may neglect the lowest term of the summation over

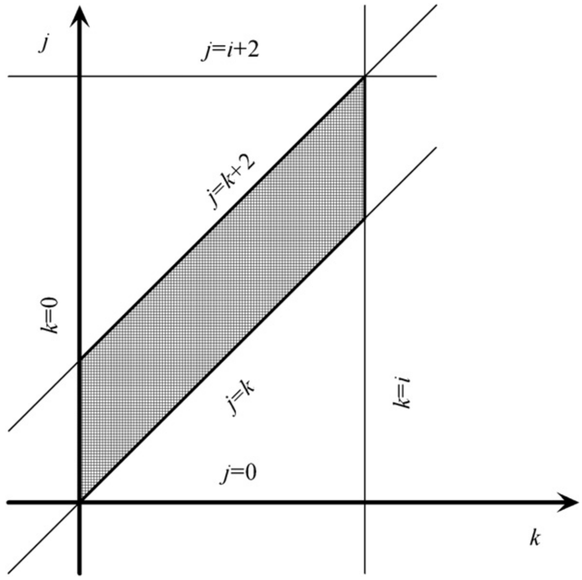

Fig. 4. The geometry of the summational transformation: $\sum_{k=0}^{i}\sum_{j=k}^{(k+2)} = \sum_{j=0}^{(i+2)}\sum_{k=\max[0,(j-2)]}^{\min[j,i]}$.

ℓ and express the limits of the ℓ summation accordingly as has been done in what follows.

Lastly, one needs to extract the $(\mathscr{g}^2)$ summation. Shifting the η index yields:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}[L_{12}(0)-L_{12}(\chi)]$$
$$=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathscr{g}^2\right)\sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^{\ell}(\chi)\right]\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}\sum_{\eta=i}^{(p+q+2-i)}\left(\mathscr{g}^2\right)^{\eta}\frac{(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\frac{8^i(-1)^{(\eta-i)}}{(\eta-i)!(p+q+2-i-\eta)!}\frac{2^{2\eta}}{4^{(p+q+2)}}\frac{(\eta+1)!}{(2(\eta+1))!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}\frac{(-1)^k}{(k)!(i-k)!}$$
$$\times\left\{\left[(p+q+1-i-\eta)(p+q+2-i-\eta)-\frac{1}{2}(\eta-i)(\eta-i-1)\right]\delta_{k,\ell}\right.$$
$$\left.+[2(\eta-i)(p+q+2-i-\eta)]\delta_{k,(\ell-1)}+\left[\frac{3}{2}(\eta-i)(\eta-i-1)\right]\delta_{k,(\ell-2)}\right\},\tag{55}$$

where it can be seen from Fig. 6 that:

$$\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}\sum_{\eta=i}^{(p+q+2-i)}=\sum_{\eta=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}\sum_{i=\max[0,(\ell-2)]}^{\min[p,q,\eta,(p+q+2-\eta)]}.\tag{56}$$

Thus:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}[L_{12}(0)-L_{12}(\chi)]=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]\,M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathscr{g}^2\right)\sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^{\ell}(\chi)\right]\sum_{\eta=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}\left(\mathscr{g}^2\right)^{\eta}$$
$$\times\sum_{i=\max[0,(\ell-2)]}^{\min[p,q,\eta,(p+q+2-\eta)]}\frac{(p+q-2i)!}{(p-i)!(q-i)!}\frac{8^i(-1)^{(\eta-i)}}{(\eta-i)!(p+q+2-i-\eta)!}\frac{2^{2\eta}}{4^{(p+q+2)}}\frac{(\eta+1)!}{(2(\eta+1))!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}$$
$$\times\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}\frac{(-1)^k}{(k)!(i-k)!}\left\{\left[(p+q+1-i-\eta)(p+q+2-i-\eta)-\frac{1}{2}(\eta-i)(\eta-i-1)\right]\delta_{k,\ell}\right.$$
$$\left.+[2(\eta-i)(p+q+2-i-\eta)]\delta_{k,(\ell-1)}+\left[\frac{3}{2}(\eta-i)(\eta-i-1)\right]\delta_{k,(\ell-2)}\right\},\tag{57}$$

and one need only let $\eta\to r$ to obtain the coefficient of $(\mathscr{g}^2)^r$, i.e.:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}[L_{12}(0)-L_{12}(\chi)]=\frac{2}{3}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^pt^q]$$
$$\times M_2^{(p+1)}M_1^{(q+1)}\exp\left(-\mathscr{g}^2\right)\sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^{\ell}(\chi)\right]$$
$$\times\sum_{r=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}\left(\mathscr{g}^2\right)^r\sum_{i=\max[0,(\ell-2)]}^{\min[p,q,r,(p+q+2-r)]}$$
$$\times\frac{(p+q-2i)!}{(p-i)!(q-i)!}\frac{8^i(-1)^{(r+i)}}{(r-i)!(p+q+2-i-r)!}\frac{2^{2r}}{4^{(p+q+2)}}$$
$$\times\frac{(r+1)!}{(2(r+1))!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}\frac{(-1)^k}{(k)!(i-k)!}$$
$$\times\left\{\left[(p+q+1-i-r)(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\right]\delta_{k,\ell}\right.$$
$$+[2(r-i)(p+q+2-i-r)]\delta_{k,(\ell-1)}$$
$$\left.+\left[\frac{3}{2}(r-i)(r-i-1)\right]\delta_{k,(\ell-2)}\right\}.\tag{58}$$

Following the above extraction of the various summations in the indicated order, and after integration of the coefficient of $[s^pt^q]$ over ε and the directions of $\mathscr{g}$, one has that:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1,S_{5/2}^{(q)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}=\frac{16}{3}M_2^{(p+1)}M_1^{(q+1)}$$
$$\times\sum_{\ell=1}^{(\min[p,q]+2)}\sum_{r=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}B_{pqr\ell}\,\Omega_{12}^{(\ell)}(r),\tag{59}$$

where:

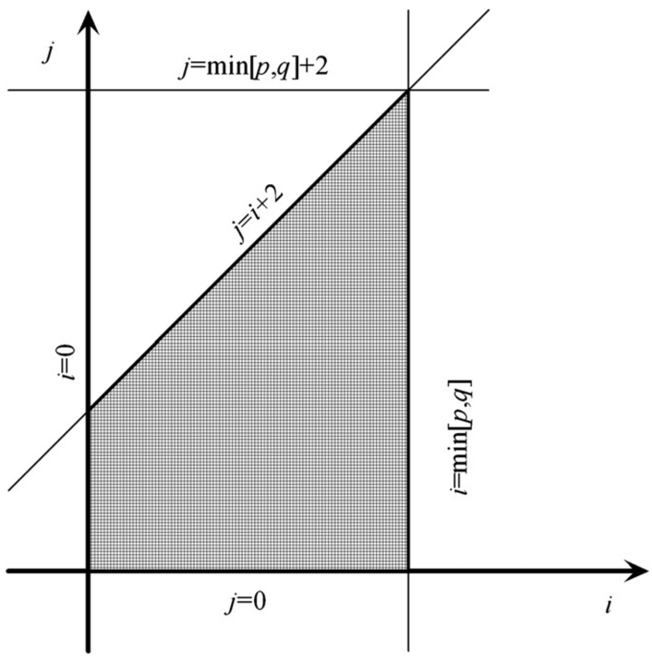

Fig. 5. The geometry of the summational transformation:

$$\sum_{i=0}^{\min[p,q]}\sum_{j=0}^{(i+2)} = \sum_{j=0}^{(\min[p,q]+2)}\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}.$$

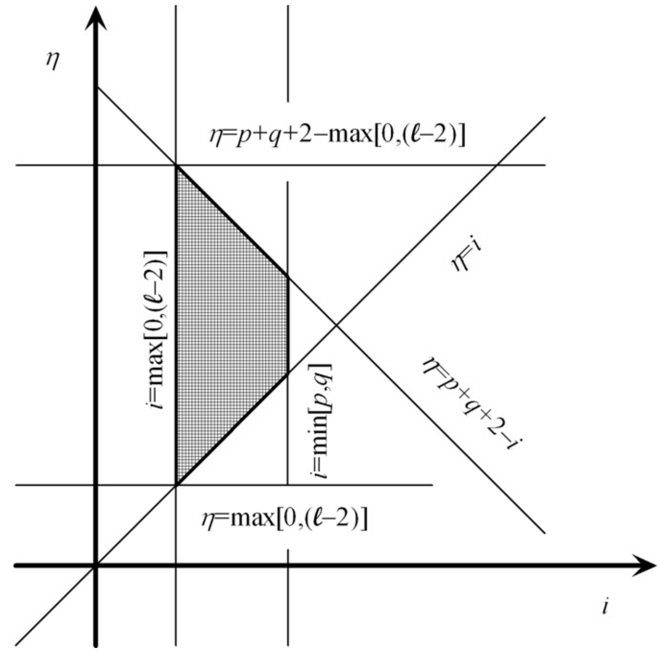

Fig. 6. The geometry of the summational transformation:

$$\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}\sum_{\eta=i}^{(p+q+2-i)} = \sum_{\eta=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}\sum_{i=\max[0,(\ell-2)]}^{\min[p,q,\eta,(p+q+2-\eta)]}.$$

$$\begin{aligned}B_{pqr\ell} = &\sum_{i=\max[0,(\ell-2)]}^{\min[p,q,r,(p+q+2-r)]}\frac{(p+q-2i)!}{(p-i)!(q-i)!}\\ &\times\frac{8^i(-1)^{(r+i)}}{(r-i)!(p+q+2-i-r)!}\frac{2^{2r}}{4^{(p+q+2)}}\frac{(r+1)!}{(2(r+1))!}\\ &\times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}\frac{(-1)^k}{(k)!(i-k)!}\\ &\times\Bigg\{\bigg[(p+q+1-i-r)(p+q+2-i-r)\\ &-\frac{1}{2}(r-i)(r-i-1)\bigg]\delta_{k,\ell}\\ &+[2(r-i)(p+q+2-i-r)]\delta_{k,(\ell-1)}\\ &+\left[\frac{3}{2}(r-i)(r-i-1)\right]\delta_{k,(\ell-2)}\Bigg\}.\end{aligned} \tag{60}$$

Here, we note that some simplification may be achieved in the above expressions by taking advantage of the fact that $\ell \geq 1$ implies that one generally has $\max[0,(\ell-2)] = (\ell-2)$ except for a limited number of specific cases that occur only when $\ell = 1$ and which must be excluded from the summations. These cases include when $r=(-1)$ and $r=(p+q+3)$, when $i=(-1)$, and when $k=(-1)$. These cases may be excluded from the various summations via the appropriate inclusion of Kronecker deltas in either Eq. (59) or Eq. (60). Thus, it is equivalent to write:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12} = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)}\sum_{r=(\ell-2)}^{(p+q+4-\ell)}B''_{pqr\ell}\,\Omega_{12}^{(\ell)}(r), \tag{61}$$

with:

$$\begin{aligned}B''_{pqr\ell} = &M_2^{(p+1)}M_1^{(q+1)}\left(1-\delta_{r,(-1)}\right)\left(1-\delta_{r,(p+q+3)}\right)\\ &\times\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]}\frac{(p+q-2i)!}{(p-i)!(q-i)!}\left(1-\delta_{i,(-1)}\right)\\ &\times\frac{8^i(-1)^{(r+i)}}{(r-i)!(p+q+2-i-r)!}\frac{2^{2r}}{4^{(p+q+2)}}\frac{(r+1)!}{(2(r+1))!}\\ &\times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{k=(\ell-2)}^{\min[\ell,i]}\frac{(-1)^k}{(k)!(i-k)!}\\ &\times\Bigg\{\bigg[(p+q+1-i-r)(p+q+2-i-r)\\ &-\frac{1}{2}(r-i)(r-i-1)\bigg]\delta_{k,\ell}\\ &+[2(r-i)(p+q+2-i-r)]\delta_{k,(\ell-1)}\\ &+\left[\frac{3}{2}(r-i)(r-i-1)\right]\delta_{k,(\ell-2)}\Bigg\}\left(1-\delta_{k,(-1)}\right).\end{aligned} \tag{62}$$

Further, when $r = (\ell-2)$ it follows that $i=k=r$ and one has that the last term in Eq. (62) is 0. Similarly, when $r = (\ell-1)$ it follows that one has either $i = k = (\ell-2)$ or $i=r$ with $k = i = r = (\ell-1)$ or $k = (\ell-2)$. All of these possibilities again imply that the last term in Eq. (62) is 0 and, thus, the two lowest terms in the r summation may be omitted, i.e. one may write Eq. (61) as:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12} = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)}\sum_{r=\ell}^{(p+q+4-\ell)}B''_{pqr\ell}\,\Omega_{12}^{(\ell)}(r), \tag{63}$$

which, makes the term $(1-\delta_{r,(-1)})$ in Eq. (62) redundant such that it may be omitted.

5. Derivation of a summational representation for the L_1 bracket integral

Next, one considers the L_1 bracket integral of Eq. (14):

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}. \tag{64}$$

From Chapman and Cowling [1] one has that this bracket integral may be expressed as the coefficient of $[s^p t^q]$ in the expansion of:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3}\int\int\int [L_1(0)-L_1(\chi)]gb\mathrm{d}b\mathrm{d}\varepsilon\mathrm{d}g, \tag{65}$$

where from Eq. (26) one has:

$$\begin{aligned}\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)=&\frac{2}{3}\exp\left(-g^2\right)\\ &\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}s^it^i\left\{M_1^2+M_2^2+2M_1M_2\cos(\chi)\right\}^i\frac{\left(g^2\right)^i}{(i)!}\\ &\times\left\{[M_2+(M_1-M_2)t]s+M_2t\right\}^n\left\{M_1^2(n+1)(n+2)S_{i+1/2}^{(n+2)}\left(g^2\right)\right.\\ &+2(n+1)M_1[M_1+M_2\cos(\chi)]g^2S_{i+3/2}^{(n+1)}\left(g^2\right)\\ &\left.+\left\{[M_1+M_2\cos(\chi)]^2-\frac{1}{2}M_2^2\left[1-\cos^2(\chi)\right]\right\}g^4S_{i+5/2}^{(n)}\left(g^2\right)\right\}.\end{aligned} \tag{66}$$

Again, the Sonine polynomials are consolidated using the definition of Eq. (11) such that:

$$\begin{aligned}&\left\{M_1^2(n+1)(n+2)S_{i+1/2}^{(n+2)}\left(g^2\right)\right.\\ &+2(n+1)M_1[M_1+M_2\cos(\chi)]g^2S_{i+3/2}^{(n+1)}\left(g^2\right)\\ &\left.+\left\{[M_1+M_2\cos(\chi)]^2-\frac{1}{2}M_2^2\left[1-\cos^2(\chi)\right]\right\}g^4S_{i+5/2}^{(n)}\left(g^2\right)\right\}\\ =&\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^\eta}{(\eta)!(n+2-\eta)!}\frac{4^\eta}{4^{(n+2)}}\\ &\times\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\\ &\times\left\{\left[(n+1-\eta)(n+2-\eta)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\right.\\ &-[2M_1M_2\eta(n+2-\eta)]\cos(\chi)\\ &\left.+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\},\end{aligned} \tag{67}$$

so that one has:

$$\begin{aligned}\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)=&\frac{2}{3}\exp\left(-g^2\right)\\ &\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}s^it^iM_1^iM_2^i\frac{2^i}{(i)!}[F+\cos(\chi)]^iM_2^n\\ &\times\{[1+Gt]s+t\}^n\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}\frac{4^\eta}{4^{(n+2)}}\\ &\times\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\\ &\times\left\{\left[(n+1-\eta)(n+2-\eta)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\right.\\ &-[2M_1M_2\eta(n+2-\eta)]\cos(\chi)\\ &\left.+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\},\end{aligned} \tag{68}$$

where $F=(M_1^2+M_2^2)/(2M_1M_2)$ and $G=(M_1-M_2)/M_2$.

To extract the summation over s, one first substitutes the binomial expansion:

$$\{[1+Gt]s+t\}^n=\sum_{j=0}^{n}\binom{n}{j}[1+Gt]^js^jt^{(n-j)}, \tag{69}$$

such that:

$$\begin{aligned}\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)=&\frac{2}{3}\exp\left(-g^2\right)\\ &\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{n}s^{(i+j)}t^{(i+n-j)}\frac{2^i}{(i)!}M_1^iM_2^iM_2^n\\ &\times[F+\cos(\chi)]^i\binom{n}{j}[1+Gt]^j\\ &\times\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}\frac{4^\eta}{4^{(n+2)}}\\ &\times\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\\ &\times\left\{\left[(n+1-\eta)(n+2-\eta)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\right.\\ &\left.-[2M_1M_2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}.\end{aligned} \tag{70}$$

Then, shifting the j index one has that:

$$\begin{aligned}\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)=&\frac{2}{3}\exp\left(-g^2\right)\\ &\times\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=i}^{(n+i)}s^jt^{(2i+n-j)}\frac{2^i}{(i)!}M_1^iM_2^iM_2^n\\ &\times[F+\cos(\chi)]^i\binom{n}{(j-i)}[1+Gt]^{(j-i)}\\ &\times\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}\frac{4^\eta}{4^{(n+2)}}\\ &\times\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\\ &\times\left\{\left[(n+1-\eta)(n+2-\eta)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\right.\\ &\left.-[2M_1M_2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}.\end{aligned} \tag{71}$$

As before, from Figs. 1 and 2, one can apply the expression from Eq. (41) such that:

$$\begin{aligned}\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)=&\frac{2}{3}\exp\left(-g^2\right)\\ &\times\sum_{j=0}^{\infty}s^j\sum_{i=0}^{j}\sum_{n=(j-i)}^{\infty}t^{(2i+n-j)}\frac{2^i}{(i)!}M_1^iM_2^iM_2^n\\ &\times[F+\cos(\chi)]^i\binom{n}{(j-i)}[1+Gt]^{(j-i)}\\ &\times\sum_{\eta=0}^{(n+2)}\frac{(-1)^\eta\left(g^2\right)^{(\eta+i)}}{(\eta)!(n+2-\eta)!}\frac{4^\eta}{4^{(n+2)}}\\ &\times\frac{(2(i+n+3))!}{(i+n+3)!}\frac{(i+\eta+1)!}{(2(i+\eta+1))!}\\ &\times\left\{\left[(n+1-\eta)(n+2-\eta)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\right.\\ &\left.-[2M_1M_2\eta(n+2-\eta)]\cos(\chi)+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\},\end{aligned} \tag{72}$$

and one need only let $j \to p$ to obtain the coefficient of s^p, i.e.:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathcal{g}^2\right) \times \sum_{p=0}^{\infty} s^p \sum_{i=0}^{p} \sum_{n=(p-i)}^{\infty} t^{(2i+n-p)} \frac{2^i}{(i)!} M_1^i M_2^i M_2^n \times [F+\cos(\chi)]^i \binom{n}{(p-i)} [1+Gt]^{(p-i)} \times \sum_{\eta=0}^{(n+2)} \frac{(-1)^\eta (\mathcal{g}^2)^{(\eta+i)}}{(\eta)!(n+2-\eta)!} \frac{4^\eta}{4^{(n+2)}} \times \frac{(2(i+n+3))!}{(i+n+3)!} \frac{(i+\eta+1)!}{(2(i+\eta+1))!} \times \left\{\left[(n+1-\eta)(n+2-\eta)M_1^2 - \frac{1}{2}\eta(\eta-1)M_2^2\right] - [2M_1M_2\eta(n+2-\eta)]\cos(\chi) + \left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}. \tag{73}$$

Now, to extract the summation over t, one substitutes the binomial expansion:

$$[1+Gt]^{(p-i)} = \sum_{w=0}^{(p-i)} \binom{(p-i)}{w} G^w t^w, \tag{74}$$

such that:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathcal{g}^2\right) \sum_{p=0}^{\infty} s^p \sum_{i=0}^{p} \sum_{w=0}^{(p-i)} \sum_{n=(p-i)}^{\infty} t^{(2i+n-p+w)} M_1^i M_2^i M_2^n \frac{2^i}{(i)!}[F+\cos(\chi)]^i \binom{n}{(p-i)} \binom{(p-i)}{w} G^w \times \sum_{\eta=0}^{(n+2)} \frac{(-1)^\eta (\mathcal{g}^2)^{(\eta+i)}}{(\eta)!(n+2-\eta)!} \frac{4^\eta}{4^{(n+2)}} \frac{(2(i+n+3))!}{(i+n+3)!} \frac{(i+\eta+1)!}{(2(i+\eta+1))!} \left\{\left[(n+1-\eta)(n+2-\eta)M_1^2 - \frac{1}{2}\eta(\eta-1)M_2^2\right] - [2M_1M_2\eta(n+2-\eta)]\cos(\chi) + \left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}. \tag{75}$$

Then, after shifting the n index, one has:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathcal{g}^2\right) \sum_{p=0}^{\infty} s^p \times \sum_{i=0}^{p} \sum_{w=0}^{(p-i)} \sum_{n=(w+i)}^{\infty} t^n M_1^i M_2^i M_2^{(p+n-2i-w)} \times \frac{2^i}{(i)!}[F+\cos(\chi)]^i \binom{(p+n-2i-w)}{(p-i)} \binom{(p-i)}{w} \times G^w \sum_{\eta=0}^{(p+n+2-2i-w)} \frac{(-1)^\eta (\mathcal{g}^2)^{(\eta+i)}}{(\eta)!(p+n+2-2i-\eta-w)!} \times \frac{4^\eta}{4^{(p+n+2-2i-w)}} \frac{(2(p+n+3-i)-2w)!}{(p+n+3-i-w)!} \times \frac{(i+\eta+1)!}{(2i+2\eta+2)!} \left\{\left[(p+n-2i+1-\eta-w) \times (p+n-2i+2-\eta-w)M_1^2 - \frac{1}{2}\eta(\eta-1)M_2^2\right] - [2M_1M_2\eta(p+n+2-2i-\eta-w)]\cos(\chi) + \left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}. \tag{76}$$

From Figs. 7 and 3 one has that:

$$\sum_{i=0}^{p} \sum_{w=0}^{(p-i)} \sum_{n=(w+i)}^{\infty} = \sum_{n=0}^{\infty} \sum_{i=0}^{\min[p,n]} \sum_{w=0}^{\min[(p-i),(n-i)]}, \tag{77}$$

such that:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathcal{g}^2\right) \sum_{p=0}^{\infty} s^p \sum_{n=0}^{\infty} t^n \times \sum_{i=0}^{\min[p,n]} \sum_{w=0}^{\min[(p-i),(n-i)]} M_1^i M_2^i M_2^{(p+n-2i-w)} \times \frac{2^i}{(i)!}[F+\cos(\chi)]^i \binom{(p+n-2i-w)}{(p-i)} \binom{(p-i)}{w} \times G^w \sum_{\eta=0}^{(p+n+2-2i-w)} \frac{(-1)^\eta (\mathcal{g}^2)^{(\eta+i)}}{(\eta)!(p+n+2-2i-\eta-w)!} \times \frac{4^\eta}{4^{(p+n+2-2i-w)}} \frac{(2(p+n+3-i)-2w)!}{(p+n+3-i-w)!} \times \frac{(i+\eta+1)!}{(2i+2\eta+2)!} \left\{\left[(p+n-2i+1-\eta-w) \times (p+n-2i+2-\eta-w)M_1^2 - \frac{1}{2}\eta(\eta-1)M_2^2\right] - [2M_1M_2\eta(p+n+2-2i-\eta-w)]\cos(\chi) + \left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}, \tag{78}$$

and one need only let $n \to q$ to obtain the coefficient of t^q, i.e.:

$$\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathcal{g}^2\right) \sum_{p=0}^{\infty} s^p \sum_{q=0}^{\infty} t^q \sum_{i=0}^{\min[p,q]} \times \sum_{w=0}^{\min[(p-i),(q-i)]} M_1^i M_2^i M_2^{(p+q-2i-w)} \frac{2^i}{(i)!}[F+\cos(\chi)]^i \times \binom{(p+q-2i-w)}{(p-i)} \binom{(p-i)}{w} G^w \sum_{\eta=0}^{(p+q+2-2i-w)} \times \frac{(-1)^\eta (\mathcal{g}^2)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta-w)!} \frac{4^\eta}{4^{(p+q+2-2i-w)}} \times \frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!} \frac{(i+\eta+1)!}{(2i+2\eta+2)!} \times \left\{\left[(p+q-2i+1-\eta-w)(p+q-2i+2-\eta-w)M_1^2 - \frac{1}{2}\eta(\eta-1)M_2^2\right] - [2M_1M_2\eta(p+q+2-2i-\eta-w)]\cos(\chi) + \left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\cos^2(\chi)\right\}. \tag{79}$$

To extract the $\cos(\chi)$ summation, one must first factor out powers of the $\cos(\chi)$ from the last term as a summation over Kronecker deltas [22] in the following manner:

$$\begin{aligned}
&\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)\\
&= \frac{2}{3}\exp\left(-\mathcal{g}^2\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]\sum_{i=0}^{\min[p,q]}\sum_{w=0}^{\min[(p-i),(q-i)]}M_1^i M_2^i M_2^{(p+q-2i-w)}\\
&\times\frac{2^i}{(i)!}[F+\cos(\chi)]^i\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w\sum_{\eta=0}^{(p+q+2-2i-w)}\frac{(-1)^\eta\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta-w)!}\\
&\times\frac{4^\eta}{4^{(p+q+2-2i-w)}}\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(i+\eta+1)!}{(2i+2\eta+2)!}\sum_{j=0}^{2}\cos^j(\chi)\\
&\times\left\{\left[(p+q-2i+1-\eta-w)(p+q-2i+2-\eta-w)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\delta_{j,0}\right.\\
&\left.-[2M_1M_2\eta(p+q+2-2i-\eta-w)]\delta_{j,1}+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\delta_{j,2}\right\}.
\end{aligned}\tag{80}$$

Substitution of the binomial expansion:

$$[F+\cos(\chi)]^i=\sum_{k=0}^{i}\binom{i}{k}F^{(i-k)}\cos^k(\chi),\tag{81}$$

then yields:

$$\begin{aligned}
&\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)\\
&= \frac{2}{3}\exp\left(-\mathcal{g}^2\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]\sum_{i=0}^{\min[p,q]}\sum_{k=0}^{i}\sum_{j=0}^{2}\cos^{(j+k)}(\chi)\sum_{w=0}^{\min[(p-i),(q-i)]}\frac{2^i}{(i)!}\binom{i}{k}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)}\\
&\times\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w\sum_{\eta=0}^{(p+q+2-2i-w)}\frac{(-1)^\eta\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta-w)!}\frac{4^\eta}{4^{(p+q+2-2i-w)}}\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\\
&\times\frac{(i+\eta+1)!}{(2i+2\eta+2)!}\left\{\left[(p+q-2i+1-\eta-w)(p+q-2i+2-\eta-w)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\delta_{j,0}\right.\\
&\left.-[2M_1M_2\eta(p+q+2-2i-\eta-w)]\delta_{j,1}+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\delta_{j,2}\right\},
\end{aligned}\tag{82}$$

which, following a shift of the j index, becomes:

$$\begin{aligned}
&\left(\frac{ST}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi)\\
&= \frac{2}{3}\exp\left(-\mathcal{g}^2\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]\sum_{i=0}^{\min[p,q]}\sum_{k=0}^{i}\sum_{j=k}^{(k+2)}\cos^{j}(\chi)\sum_{w=0}^{\min[(p-i),(q-i)]}\frac{2^i}{(i)!}\binom{i}{k}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)}\\
&\times\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w\sum_{\eta=0}^{(p+q+2-2i-w)}\frac{(-1)^\eta\left(\mathcal{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta-w)!}\frac{4^\eta}{4^{(p+q+2-2i-w)}}\\
&\times\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(i+\eta+1)!}{(2i+2\eta+2)!}\left\{\left[(p+q-2i+1-\eta-w)(p+q-2i+2-\eta-w)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\delta_{k,j}\right.\\
&\left.-[2M_1M_2\eta(p+q+2-2i-\eta-w)]\delta_{k,(j-1)}+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\delta_{k,(j-2)}\right\}.
\end{aligned}\tag{83}$$

Again, using Eq. (52) obtained from Figs. 4 and 5 yields:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathscr{g}^2\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]\sum_{j=0}^{(\min[p,q]+2)}\cos^j(\chi)\sum_{i=\max[0,(j-2)]}^{\min[p,q]}\sum_{k=\max[0,(j-2)]}^{\min[j,i]}$$
$$\times\sum_{w=0}^{\min[(p-i),(q-i)]}\frac{2^i}{(i)!}\binom{i}{k}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)}\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w\sum_{\eta=0}^{(p+q+2-2i-w)}$$
$$\times\frac{(-1)^\eta\left(\mathscr{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta-w)!}\frac{4^\eta}{4^{(p+q+2-2i-w)}}\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(i+\eta+1)!}{(2i+2\eta+2)!}$$
$$\times\left\{\left[(p+q-2i+1-\eta-w)(p+q-2i+2-\eta-w)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\delta_{k,j}\right.$$
$$\left.-[2M_1M_2\eta(p+q+2-2i-\eta-w)]\delta_{k,(j-1)}+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\delta_{k,(j-2)}\right\}, \quad (84)$$

and one need only let $j\to\ell$ to obtain the coefficient of $\cos^\ell(\chi)$, i.e.:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}L_1(\chi) = \frac{2}{3}\exp\left(-\mathscr{g}^2\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]$$
$$\times\sum_{\ell=0}^{(\min[p,q]+2)}\cos^\ell(\chi)\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}$$
$$\times\sum_{w=0}^{\min[(p-i),(q-i)]}\frac{2^i}{(i)!}\binom{i}{k}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)}$$
$$\times\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w\sum_{\eta=0}^{(p+q+2-2i-w)}$$
$$\times\frac{(-1)^\eta\left(\mathscr{g}^2\right)^{(\eta+i)}}{(\eta)!(p+q+2-2i-\eta-w)!}\frac{4^\eta}{4^{(p+q+2-2i-w)}}$$
$$\times\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(i+\eta+1)!}{(2i+2\eta+2)!}$$
$$\times\left\{\left[(p+q-2i+1-\eta-w)\right.\right.$$
$$\left.\times(p+q-2i+2-\eta-w)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\delta_{k,\ell}$$
$$-[2M_1M_2\eta(p+q+2-2i-\eta-w)]\delta_{k,(\ell-1)}$$
$$\left.+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\delta_{k,(\ell-2)}\right\}. \quad (85)$$

As before, note that the full integration involves the difference $[L_1(0)-L_1(\chi)]$ which yields terms containing $[1-\cos^\ell(\chi)]$. When $\ell = 0$, this quantity is identically zero and, hence, without loss of generality, one may again neglect the lowest term of the summation over ℓ and express the limits of the ℓ summation accordingly. Thus:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}[L_1(0)-L_1(\chi)]$$
$$= \frac{2}{3}\exp\left(-\mathscr{g}^2\right)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]\sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^\ell(\chi)\right]\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}\sum_{w=0}^{\min[(p-i),(q-i)]}\sum_{\eta=0}^{(p+q+2-2i-w)}\left(\mathscr{g}^2\right)^{(\eta+i)}\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}$$
$$\times\frac{2^i}{(i)!}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)}\binom{i}{k}\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w\frac{(-1)^\eta}{(\eta)!(p+q+2-2i-\eta-w)!}\frac{4^\eta}{4^{(p+q+2-2i-w)}}$$
$$\times\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(i+\eta+1)!}{(2i+2\eta+2)!}\left\{\left[(p+q-2i+1-\eta-w)(p+q-2i+2-\eta-w)M_1^2-\frac{1}{2}\eta(\eta-1)M_2^2\right]\delta_{k,\ell}\right.$$
$$\left.-[2M_1M_2\eta(p+q+2-2i-\eta-w)]\delta_{k,(\ell-1)}+\left[\frac{3}{2}M_2^2\eta(\eta-1)\right]\delta_{k,(\ell-2)}\right\}. \quad (86)$$

Lastly, one needs to extract the $(\mathscr{g}^2)$ summation. Shifting the η index yields:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2}\pi^{-3/2}[L_1(0)-L_1(\chi)] = \frac{2}{3}\exp\left(-\mathscr{g}^2\right)$$
$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q]\sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^\ell(\chi)\right]\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]}$$
$$\times\sum_{w=0}^{\min[(p-i),(q-i)]}\sum_{\eta=i}^{(p+q+2-i-w)}\left(\mathscr{g}^2\right)^\eta\sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]}$$
$$\times\frac{2^i}{(i)!}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)}$$
$$\times\binom{i}{k}\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w$$
$$\times\frac{(-1)^{(\eta-i)}}{(\eta-i)!(p+q+2-i-\eta-w)!}\frac{4^{(\eta+i)}}{4^{(p+q+2-w)}}$$
$$\times\frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(\eta+1)!}{(2\eta+2)!}$$
$$\times\left\{\left[(p+q-i+1-\eta-w)(p+q-i+2-\eta-w)M_1^2\right.\right.$$
$$\left.-\frac{1}{2}(\eta-i)(\eta-i-1)M_2^2\right]\delta_{k,\ell}$$
$$-[2M_1M_2(\eta-i)(p+q+2-i-\eta-w)]\delta_{k,(\ell-1)}$$
$$\left.+\left[\frac{3}{2}M_2^2(\eta-i)(\eta-i-1)\right]\delta_{k,(\ell-2)}\right\}. \quad (87)$$

From Figs. 8 and 6 one has that:

$$\sum_{i=\max[0,(\ell-2)]}^{\min[p,q]} \sum_{w=0}^{\min[(p-i),(q-i)]} \sum_{\eta=i}^{(p+q+2-i-w)} = \sum_{\eta=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])} \times \sum_{i=\max[0,(\ell-2)]}^{\min[p,q,\eta,(p+q+2-\eta)]} \sum_{w=0}^{\min[(p-i),(q-i),(p+q+2-i-\eta)]}, \qquad (88)$$

such that:

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2} \pi^{-3/2}[L_1(0)-L_1(\chi)] = \frac{2}{3}\exp\left(-\mathscr{g}^2\right) \times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q] \sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^{\ell}(\chi)\right] \times \sum_{\eta=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}\left(\mathscr{g}^2\right)^{\eta} \sum_{i=\max[0,(\ell-2)]}^{\min[p,q,\eta,(p+q+2-\eta)]} \times \sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]} \sum_{w=0}^{\min[(p-i),(q-i),(p+q+2-i-\eta)]} \times \frac{2^i}{(i)!}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)} \times \binom{i}{k}\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w \times \frac{(-1)^{(\eta-i)}}{(\eta-i)!(p+q+2-i-\eta-w)!}\frac{4^{(\eta+i)}}{4^{(p+q+2-w)}} \times \frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(\eta+1)!}{(2\eta+2)!} \times \left\{\left[(p+q-i+1-\eta-w)(p+q-i+2-\eta-w)M_1^2 - \frac{1}{2}(\eta-i)(\eta-i-1)M_2^2\right]\delta_{k,\ell} - [2M_1M_2(\eta-i)(p+q+2-i-\eta-w)]\delta_{k,(\ell-1)} + \left[\frac{3}{2}M_2^2(\eta-i)(\eta-i-1)\right]\delta_{k,(\ell-2)}\right\}, \qquad (89)$$

and one need only let $\eta\to r$ to obtain the coefficient of $(\mathscr{g}^2)^r$, i.e.:

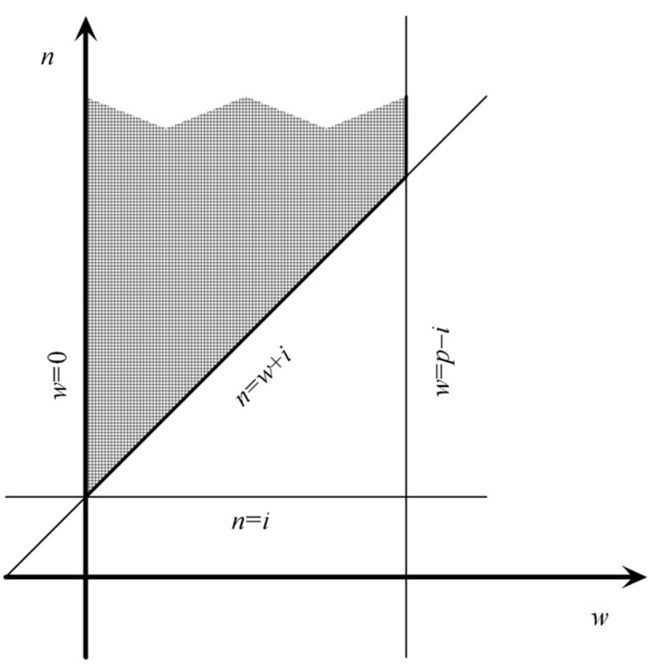

Fig. 7. The geometry of the summational transformation: $\sum_{w=0}^{(p-i)}\sum_{n=(w+i)}^{\infty} = \sum_{n=i}^{\infty}\sum_{w=0}^{\min[(p-i),(n-i)]}$.

$$\left(\frac{\mathrm{ST}}{st}\right)^{7/2} \pi^{-3/2}[L_1(0)-L_1(\chi)] = \frac{2}{3}\exp\left(-\mathscr{g}^2\right) \times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}[s^p t^q] \sum_{\ell=1}^{(\min[p,q]+2)}\left[1-\cos^{\ell}(\chi)\right] \times \sum_{r=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])}\left(\mathscr{g}^2\right)^{r} \sum_{i=\max[0,(\ell-2)]}^{\min[p,q,r,(p+q+2-r)]} \times \sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]} \sum_{w=0}^{\min[(p-i),(q-i),(p+q+2-i-r)]} \times \frac{2^i}{(i)!}M_1^i M_2^i M_2^{(p+q-2i-w)}F^{(i-k)} \times \binom{i}{k}\binom{(p+q-2i-w)}{(p-i)}\binom{(p-i)}{w}G^w \times \frac{(-1)^{(r-i)}}{(r-i)!(p+q+2-i-r-w)!}\frac{4^{(r+i)}}{4^{(p+q+2-w)}} \times \frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{(r+1)!}{(2r+2)!} \times \left\{\left[(p+q-i+1-r-w)(p+q-i+2-r-w)M_1^2 - \frac{1}{2}(r-i)(r-i-1)M_2^2\right]\delta_{k,\ell} - [2M_1M_2(r-i)(p+q+2-i-r-w)]\delta_{k,(\ell-1)} + \left[\frac{3}{2}M_2^2(r-i)(r-i-1)\right]\delta_{k,(\ell-2)}\right\}. \qquad (90)$$

Now, after integration of the coefficient of $[s^p t^q]$ over ε and the directions of $\mathscr{g}$, one has:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)} \sum_{r=\max[0,(\ell-2)]}^{(p+q+2-\max[0,(\ell-2)])} B'_{pqr\ell}\,\Omega_{12}^{(\ell)}(r), \qquad (91)$$

where:

$$B'_{pqr\ell} = \sum_{i=\max[0,(\ell-2)]}^{\min[p,q,r,(p+q+2-r)]} \sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]} \frac{2^{2r}}{4^{(p+q+2)}} \times \sum_{w=0}^{(\min[p,q,(p+q+2-r)]-i)} \frac{8^i(p+q-2i-w)!}{(p-i-w)!(q-i-w)!} \times \frac{(-1)^{(r+i)}}{(r-i)!(p+q+2-i-r-w)!}\frac{(r+1)!}{(2r+2)!} \times \frac{(2(p+q+3-i)-2w)!}{(p+q+3-i-w)!}\frac{1}{(k)!(i-k)!}\frac{1}{(w)!} \times 2^{(2w-2)}F^{(i-k)}G^w M_1^i M_2^i M_2^{(p+q-2i-w)}4 \times \left\{\left[(p+q+1-i-r-w)(p+q+2-i-r-w)M_1^2 - \frac{1}{2}(r-i)(r-i-1)M_2^2\right]\delta_{k,\ell} - [2M_1M_2(r-i)(p+q+2-i-r-w)]\delta_{k,(\ell-1)} + \left[\frac{3}{2}M_2^2(r-i)(r-i-1)\right]\delta_{k,(\ell-2)}\right\}. \qquad (92)$$

Now, one may introduce Pochhammer's notation [23,24] which is defined as:

$$(z)_n \equiv \frac{\Gamma(z+n)}{\Gamma(z)} = \frac{(z+n-1)!}{(z-1)!}, \qquad (93)$$

such that:

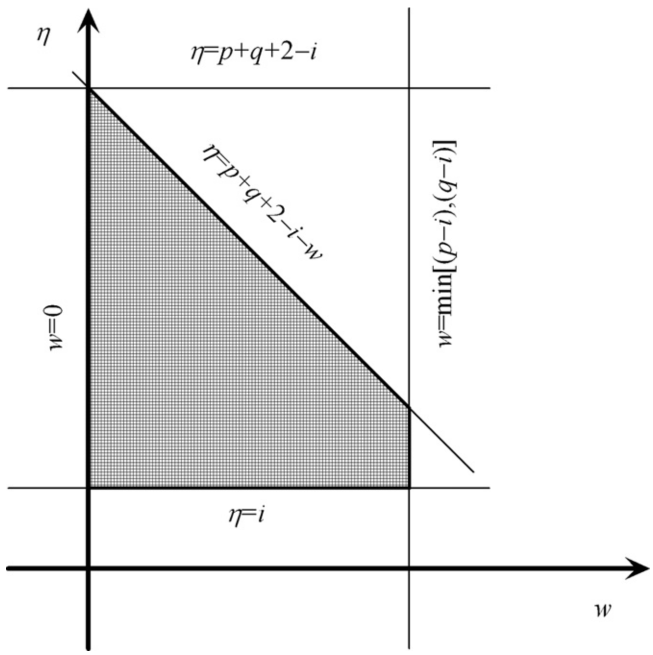

Fig. 8. The geometry of the summational transformation: $\sum_{w=0}^{\min[(p-i),(q-i)]}\sum_{\eta=i}^{(p+q+2-i-w)} = \sum_{\eta=i}^{(p+q+2-i)}\sum_{w=0}^{\min[(p-i),(q-i),(p+q+2-i-\eta)]}$.

$$(z)! = \frac{(z+n)!}{(z+1)_n}. \tag{94}$$

Using this notation, one may write:

$$(z-w)! = \frac{(z)!}{(z-w+1)_w}, \tag{95}$$

and:

$$(2z-2w)! = \frac{(2z)!}{(2z-2w+1)_w(2z-w+1)_w}, \tag{96}$$

such that $B'_{pqr\ell}$ may be expressed as:

$$\begin{aligned}B'_{pqr\ell} = {} & \sum_{i=\max[0,(\ell-2)]}^{\min[p,q,r,(p+q+2-r)]}\ \sum_{k=\max[0,(\ell-2)]}^{\min[\ell,i]} \frac{2^{2r}}{4^{(p+q+2)}}\frac{8^i(p+q-2i)!}{(p-i)!(q-i)!}\frac{(-1)^{(r+i)}}{(r-i)!(p+q+2-i-r)!}\\ & \times\frac{(r+1)!}{(2r+2)!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{w=0}^{(\min[p,q,(p+q+2-r)]-i)}\frac{(p+1-i-w)_w(q+1-i-w)_w}{(w)!(p+q+1-2i-w)_w}\frac{F^{(i-k)}}{(k)!(i-k)!}\\ & \times\frac{(p+q+3-i-r-w)_w}{(2(p+q+3-i)-2w+1)_w}2^{(2w-2)}G^w\frac{(p+q+4-i-w)_w}{(2(p+q+3-i)-w+1)_w}\\ & \times M_1^iM_2^iM_2^{(p+q-2i-w)}4\Big\{\Big[(p+q+1-i-r-w)(p+q+2-i-r-w)M_1^2-\frac{1}{2}(r-i)(r-i-1)M_2^2\Big]\delta_{k,\ell}\\ & -[2M_1M_2(r-i)(p+q+2-i-r-w)]\delta_{k,(\ell-1)}+\Big[\frac{3}{2}M_2^2(r-i)(r-i-1)\Big]\delta_{k,(\ell-2)}\Big\}.\end{aligned} \tag{97}$$

Here, except for a factor of $(-1)^k$ and the mass- and w- dependencies in the last term, the terms preceding the summation over w are the same as those obtained for $B''_{pqr\ell}$ in Eq. (62) and the entire mass dependence of $B'_{pqr\ell}$ has been collected inside the summation over w. Again, using the same logic that was used to obtain $B''_{pqr\ell}$ in Eq. (62), one can introduce Kronecker deltas into Eqs. (91) and (97) and simplify the limits of the summations such that:

$$\begin{aligned}&\Big[S^{(p)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(q)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\Big]_{12}\\ &\qquad = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)}\ \sum_{r=\ell}^{(p+q+4-\ell)} B'_{pqr\ell}\,\Omega_{12}^{(\ell)}(r),\end{aligned} \tag{98}$$

and:

$$\begin{aligned}B'_{pqr\ell} = {} & \Big(1-\delta_{r,(-1)}\Big)\Big(1-\delta_{r,(p+q+3)}\Big)\\ & \times\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]}\frac{2^{2r}}{4^{(p+q+2)}}\frac{8^i(p+q-2i)!}{(p-i)!(q-i)!}\\ & \times\frac{(-1)^{(r+i)}\Big(1-\delta_{i,(-1)}\Big)}{(r-i)!(p+q+2-i-r)!}\frac{(r+1)!}{(2r+2)!}\\ & \times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{w=0}^{(\min[p,q,(p+q+2-r)]-i)}\\ & \times\frac{(p+1-i-w)_w(q+1-i-w)_w}{(w)!(p+q+1-2i-w)_w}\\ & \times\frac{(p+q+3-i-r-w)_w}{(2(p+q+3-i)-2w+1)_w}2^{(2w-2)}G^w\\ & \times\frac{(p+q+4-i-w)_w}{(2(p+q+3-i)-w+1)_w}M_1^iM_2^iM_2^{(p+q-2i-w)}\\ & \times\sum_{k=(\ell-2)}^{\min[\ell,i]}\frac{F^{(i-k)}}{(k)!(i-k)!}\Big(1-\delta_{k,(-1)}\Big)4\\ & \times\Big\{\Big[(p+q+1-i-r-w)(p+q+2-i-r-w)M_1^2\\ & \quad-\frac{1}{2}(r-i)(r-i-1)M_2^2\Big]\delta_{k,\ell}\\ & -[2M_1M_2(r-i)(p+q+2-i-r-w)]\delta_{k,(\ell-1)}\\ & +\Big[\frac{3}{2}M_2^2(r-i)(r-i-1)\Big]\delta_{k,(\ell-2)}\Big\}.\end{aligned} \tag{99}$$

Here, again, simplification of the lower limit of the r summation has made the factor $(1-\delta_{r,\ (-1)})$ redundant. Now one may consider some additional simplification of $B''_{pqr\ell}$ and $B'_{pqr\ell}$. In both cases the summations over k contain, at most, three terms corresponding to $k = (\ell-2)$, $k = (\ell-1)$, and $k = \ell$. Expanding and consolidating the k summation in Eq. (62), one has that:

$$\sum_{k=(\ell-2)}^{\min[\ell,i]} \frac{(-1)^k}{(k)!(i-k)!}\Big(1-\delta_{k,(-1)}\Big)\Big\{\Big[(p+q+1-i-r)(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\Big]\delta_{k,\ell}$$
$$+[2(r-i)(p+q+2-i-r)]\delta_{k,(\ell-1)}+\Big[\frac{3}{2}(r-i)(r-i-1)\Big]\delta_{k,(\ell-2)}\Big\}$$
$$=\frac{(-1)^{(\ell)}}{(\ell)!(i+2-\ell)!}\Big\{(i+1-\ell)(i+2-\ell)\Big[(p+q+1-i-r)(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\Big]$$
$$+\frac{3}{2}(\ell-1)\ell(r-i)(r-i-1)-2\ell(i+2-\ell)(r-i)(p+q+2-i-r)\Big\}, \tag{100}$$

such that:

$$B''_{pqr\ell} = M_2^{(p+1)}M_1^{(q+1)}\Big(1-\delta_{r,(p+q+3)}\Big)$$
$$\times\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]}\frac{2^{2r}}{4^{(p+q+2)}}\frac{8^i(p+q-2i)!}{(p-i)!(q-i)!}\frac{(-1)^{(r+i)}\Big(1-\delta_{i,(-1)}\Big)}{(r-i)!(p+q+2-i-r)!}\frac{(r+1)!}{(2r+2)!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\frac{(-1)^{(\ell)}}{(\ell)!(i+2-\ell)!}$$
$$\times\Big\{(i+1-\ell)(i+2-\ell)\Big[(p+q+1-i-r)(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\Big]+\frac{3}{2}(\ell-1)\ell(r-i)(r-i-1)$$
$$-2\ell(i+2-\ell)(r-i)(p+q+2-i-r)\Big\}. \tag{101}$$

Expanding and consolidating the k summation in Eq. (99), one has that:

$$\sum_{k=(\ell-2)}^{\min[\ell,i]}\frac{F^{(i-k)}}{(k)!(i-k)!}\Big(1-\delta_{k,(-1)}\Big)4\Big\{\Big[(p+q+1-i-r-w)(p+q+2-i-r-w)M_1^2-\frac{1}{2}(r-i)(r-i-1)M_2^2\Big]\delta_{k,\ell}$$
$$-[2M_1M_2(r-i)(p+q+2-i-r-w)]\delta_{k,(\ell-1)}+\Big[\frac{3}{2}M_2^2(r-i)(r-i-1)\Big]\delta_{k,(\ell-2)}\Big\}$$
$$=\frac{F^{(i+2-\ell)}M_1^2}{(\ell)!(i+2-\ell)!}4\Big\{\frac{3}{2}\frac{M_2^2}{M_1^2}\ell(\ell-1)(r-i)(r-i-1)-\frac{2}{F}\frac{M_2}{M_1}\ell(i+2-\ell)(r-i)(p+q+2-i-r-w)$$
$$+\frac{1}{F^2}(i+1-\ell)(i+2-\ell)\Big[(p+q+1-i-r-w)(p+q+2-i-r-w)-\frac{1}{2}\frac{M_2^2}{M_1^2}(r-i)(r-i-1)\Big]\Big\}, \tag{102}$$

such that:

$$B'_{pqr\ell} = \Big(1-\delta_{r,(p+q+3)}\Big)\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]}\frac{2^{2r}}{4^{(p+q+2)}}\frac{8^i(p+q-2i)!}{(p-i)!(q-i)!}\frac{(-1)^{(r+i)}\Big(1-\delta_{i,(-1)}\Big)}{(r-i)!(p+q+2-i-r)!}\frac{(r+1)!}{(2r+2)!}$$
$$\times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{w=0}^{(\min[p,q,(p+q+2-r)]-i)}\frac{(p+1-i-w)_w(q+1-i-w)_w}{(w)!(p+q+1-2i-w)_w}$$
$$\times\frac{(p+q+3-i-r-w)_w}{(2(p+q+3-i)-2w+1)_w}2^{(2w-2)}G^w\frac{(p+q+4-i-w)_w}{(2(p+q+3-i)-w+1)_w}M_1^iM_2^iM_2^{(p+q-2i-w)}$$
$$\times\frac{F^{(i+2-\ell)}M_1^2}{(\ell)!(i+2-\ell)!}4\Big\{\frac{3}{2}\frac{M_2^2}{M_1^2}\ell(\ell-1)(r-i)(r-i-1)-\frac{2}{F}\frac{M_2}{M_1}\ell(i+2-\ell)(r-i)(p+q+2-i-r-w)$$
$$+\frac{1}{F^2}(i+1-\ell)(i+2-\ell)\Big[(p+q+1-i-r-w)(p+q+2-i-r-w)-\frac{1}{2}\frac{M_2^2}{M_1^2}(r-i)(r-i-1)\Big]\Big\}. \tag{103}$$

In Eq. (103), the mass- and w-independent portion of $B'_{pqr\ell}$ is essentially identical to the mass-independent portion of $B''_{pqr\ell}$ in Eq. (101) except for an absent factor of $(-1)^{\ell}$ which is present in $B''_{pqr\ell}$ due to the additional factor of $(-1)^k$ noted previously and the now mass- and w-dependent term containing the Kronecker deltas from the extraction of powers of $\cos(\chi)$. Here, one can see that the total power of M contained in the combined i and w summations (or total power of 1/2 in the limit when $M_1=M_2=1/2$ where $w=0$) is simply $p+q+2$. The quantities F and G make no net contribution to this total power of M although they do adjust the specific powers of M_1 and M_2 that contribute to it such that the mass dependence of each term ends up being a polynomial in various products of $M_1^a M_2^b$ where $a+b=p+q+2$. This is the total power of M that one expects to see out of this derivation and is the same as the total power of M seen in the previous derivation of the L_{12} bracket integral expression. Thus, both derivations are consistent with the combinatorial rule for generation of the simple gas bracket integral as described in Chapman and Cowling [1] and in our previous work [18,20]. The extra factor of $(-1)^{\ell}$ that was generated in $B''_{pqr\ell}$ during the process of expanding and consolidating the k summation and which is not present in the expression for $B'_{pqr\ell}$ is, again, expected in light of the combinatorial rule for the simple gas bracket integral which is discussed next.

6. Derivation of a summational representation for the simple gas bracket integral

Lastly, the simple gas bracket integral is considered. Here, a combinatorial rule is used to generate an appropriate expression for the bracket integral from the L_{12} and L_1 bracket integrals derived above. The appropriate combinatorial rule from Chapman and Cowling [1] is:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_1 = \left\{\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12} + \left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}\right\}\Bigg|_{\substack{m_1=m_2\\ \sigma_1=\sigma_2}}. \tag{104}$$

From this, it is apparent that the use of Eqs. (101) and (103) yields:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_1 = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)}\sum_{r=\ell}^{(p+q+4-\ell)}\Omega_1^{(\ell)}(r)\left\{B''_{pqr\ell}+B'_{pqr\ell}\right\}\Big|_{m_1=m_2}, \tag{105}$$

where $\Omega_1^{(\ell)}(r)$ are the simple gas omega integrals which can be obtained directly from the definition of the omega integrals given by Eqs. (19) and (20). Since in the limit of $m_1=m_2$ one has that $M_1=M_2=1/2$ such that $w=0$, since $(M_1-M_2)^0\to 1$, and since $(z)_0=(z-1)!/(z-1)!=1$, one has from Eqs. (101) and (103) that:

$$\left\{B''_{pqr\ell}+B'_{pqr\ell}\right\}\Big|_{m_1=m_2} = \left(\frac{1}{2}\right)^{(p+q+2)}\frac{2^{2r}}{4^{(p+q+2)}}\left(1-\delta_{r,(p+q+3)}\right)\frac{(r+1)!}{(2r+2)!}\frac{\left[1+(-1)^{\ell}\right]}{(\ell)!}\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]} \times\frac{(-1)^{(r+i)}}{(p+q+2-i-r)!}\frac{(p+q-2i)!}{(p-i)!(q-i)!(r-i)!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\frac{8^i}{(i+2-\ell)!}\left(1-\delta_{i,(-1)}\right) \times\left\{(i+1-\ell)(i+2-\ell)\left[(p+q+1-i-r)(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\right] +\frac{3}{2}(\ell-1)\ell(r-i)(r-i-1)-2\ell(i+2-\ell)(r-i)(p+q+2-i-r)\right\}, \tag{106}$$

such that the simple gas bracket integrals may be expressed as:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_1 = \frac{16}{3}\sum_{\ell=2}^{(\min[p,q]+2)}\sum_{r=\ell}^{(p+q+4-\ell)}B'''_{pqr\ell}\,\Omega_1^{(\ell)}(r), \tag{107}$$

in which:

$$B'''_{pqr\ell} = \left(\frac{1}{2}\right)^{(p+q+2)}\frac{2^{2r}}{4^{(p+q+2)}} \times\frac{(r+1)!}{(2r+2)!}\frac{\left[1+(-1)^{\ell}\right]}{(\ell)!}\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]} \times\frac{(-1)^{(r+i)}}{(p+q+2-i-r)!}\frac{(p+q-2i)!}{(p-i)!(q-i)!(r-i)!} \times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\frac{8^i}{(i+2-\ell)!} \times\left\{(i+1-\ell)(i+2-\ell)\left[(p+q+1-i-r)\times(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\right] +\frac{3}{2}(\ell-1)\ell(r-i)(r-i-1) -2\ell(i+2-\ell)(r-i)(p+q+2-i-r)\right\}. \tag{108}$$

Here, as expected, a factor of $[1+(-1)^{\ell}]$ is present which results in the elimination of all of the terms associated with odd values of ℓ and generates an additional factor of 2 in all of the terms associated with even values of ℓ. This is reflected in the adjustment of the lower limit of the ℓ summation from $\ell=1$ to $\ell=2$ and the dropping of the various redundant functions containing Kronecker deltas since, with $\ell\geq 2$, one always has that $r\neq(-1)$, $r\neq(p+q+3)$ and $i\neq(-1)$.

7. Summary of the general bracket integral expressions

In summary, one has the following general summational expressions for the viscosity-related bracket integrals. First, the L_{12} bracket integrals of Eq. (13) are expressible as:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12} = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)}\sum_{r=\ell}^{(p+q+4-\ell)} B''_{pqr\ell}\,\Omega_{12}^{(\ell)}(r), \quad (109)$$

in which:

$$B''_{pqr\ell} = M_2^{(p+1)}M_1^{(q+1)}\left(1-\delta_{r,(p+q+3)}\right) \times \sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]} \frac{2^{2r}}{4^{(p+q+2)}}\frac{8^i(p+q-2i)!}{(p-i)!(q-i)!}\frac{(-1)^{(r+i)}\left(1-\delta_{i,(-1)}\right)}{(r-i)!(p+q+2-i-r)!}\frac{(r+1)!}{(2r+2)!}\frac{(2(p+q+3-i))!}{(p+q+3-i)!}$$
$$\times\frac{(-1)^{(\ell)}}{(\ell)!(i+2-\ell)!}\left\{(i+1-\ell)(i+2-\ell)\left[(p+q+1-i-r)(p+q+2-i-r)-\frac{1}{2}(r-i)(r-i-1)\right]\right.$$
$$\left.+\frac{3}{2}(\ell-1)\ell(r-i)(r-i-1)-2\ell(i+2-\ell)(r-i)(p+q+2-i-r)\right\}. \quad (110)$$

Second, the L_1 bracket integrals of Eq. (14) are expressible as:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} = \frac{16}{3}\sum_{\ell=1}^{(\min[p,q]+2)}\sum_{r=\ell}^{(p+q+4-\ell)} B'_{pqr\ell}\,\Omega_{12}^{(\ell)}(r), \quad (111)$$

in which:

$$B'_{pqr\ell} = \left(1-\delta_{r,(p+q+3)}\right)\times\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]}\frac{2^{2r}}{4^{(p+q+2)}}\frac{8^i(p+q-2i)!}{(p-i)!(q-i)!}$$
$$\times\frac{(-1)^{(r+i)}\left(1-\delta_{i,(-1)}\right)}{(r-i)!(p+q+2-i-r)!}\frac{(r+1)!}{(2r+2)!}$$
$$\times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\sum_{w=0}^{(\min[p,q,(p+q+2-r)]-i)}$$
$$\times\frac{(p+1-i-w)_w(q+1-i-w)_w}{(w)!(p+q+1-2i-w)_w}$$
$$\times\frac{(p+q+3-i-r-w)_w}{(2(p+q+3-i)-2w+1)_w}2^{(2w-2)}G^w$$
$$\times\frac{(p+q+4-i-w)_w}{(2(p+q+3-i)-w+1)_w}M_1^iM_2^iM_2^{(p+q-2i-w)}$$
$$\times\frac{F^{(i+2-\ell)}M_1^2}{(\ell)!(i+2-\ell)!}4\left\{\frac{3}{2}\frac{M_2^2}{M_1^2}\ell(\ell-1)(r-i)(r-i-1)\right.$$
$$-\frac{2}{F}\frac{M_2}{M_1}\ell(i+2-\ell)(r-i)(p+q+2-i-r-w)$$
$$+\frac{1}{F^2}(i+1-\ell)(i+2-\ell)\left[(p+q+1-i-r-w)\right.$$
$$\times(p+q+2-i-r-w)$$
$$\left.\left.-\frac{1}{2}\frac{M_2^2}{M_1^2}(r-i)(r-i-1)\right]\right\}. \quad (112)$$

(recall the definitions $F=(M_1^2+M_2^2)/(2M_1M_2)$ and $G=(M_1-M_2)/M_2)$ and third, the simple gas bracket integrals of Eq. (12), which are obtained by appropriate combinations of the L_{12} and L_1 bracket integrals, are similarly expressible as:

$$\left[S_{5/2}^{(p)}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S_{5/2}^{(q)}(C_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_1 = \frac{16}{3}\sum_{\ell=2}^{(\min[p,q]+2)}\sum_{r=\ell}^{(p+q+4-\ell)} B'''_{pqr\ell}\,\Omega_1^{(\ell)}(r), \quad (113)$$

in which:

$$B'''_{pqr\ell} = \left(\frac{1}{2}\right)^{(p+q+2)}\frac{2^{2r}}{4^{(p+q+2)}}$$
$$\times\frac{(r+1)!}{(2r+2)!}\frac{\left[1+(-1)^{\ell}\right]}{(\ell)!}\sum_{i=(\ell-2)}^{\min[p,q,r,(p+q+2-r)]}$$
$$\times\frac{(-1)^{(r+i)}}{(p+q+2-i-r)!}\frac{(p+q-2i)!}{(p-i)!(q-i)!(r-i)!}$$
$$\times\frac{(2(p+q+3-i))!}{(p+q+3-i)!}\frac{8^i}{(i+2-\ell)!}\left\{(i+1-\ell)(i+2-\ell)\right.$$
$$\times\left[(p+q+1-i-r)(p+q+2-i-r)\right.$$
$$\left.-\frac{1}{2}(r-i)(r-i-1)\right]+\frac{3}{2}(\ell-1)\ell(r-i)(r-i-1)$$
$$\left.-2\ell(i+2-\ell)(r-i)(p+q+2-i-r)\right\}. \quad (114)$$

To complete this summary of the current results, note that in Eqs. (110) and (112) one has the the gas mixture omega integrals that were defined previously by Eqs. (17) and (18) for collisions between unlike molecules in the mixture as:

$$\Omega_{12}^{(\ell)}(r) = \frac{1}{2}\sigma_{12}^2\left(\frac{2\pi kT}{m_0M_1M_2}\right)^{1/2}2\int_0^\infty\exp\left(-g^2\right)g^{(2r+3)}$$
$$\times\int_0^\pi\left[1-\cos^\ell(\chi)\right](b/\sigma_{12})\mathrm{d}(b/\sigma_{12})\mathrm{d}g, \quad (115)$$

while in Eq. (113), where collisions in the mixture are between like molecules of a single species, one has the simple gas omega integrals which were defined previously by Eqs. (19) and (20) as:

$$\Omega_1^{(\ell)}(r) = \sigma_1^2\left(\frac{\pi kT}{m_1}\right)^{1/2} 2\int_0^\infty \exp\left(-g^2\right) g^{(2r+3)} \times \int_0^\pi \left[1-\cos^\ell(\chi)\right](b/\sigma_1)\mathrm{d}(b/\sigma_1)\mathrm{d}g. \quad (116)$$

All of the other various notations employed in Eqs. (109)–(116) have been defined at various points in the body of this document.

8. Explicit expressions for the viscosity bracket integrals up to order 5

In most of the computer codes implemented to date that utilize bracket integrals in their calculations, the emphasis has been on the use of explicit bracket integral expressions up to orders 1, 2, or at most, 3. The reason for this is that the complexity of the explicit expressions has made them difficult to derive reliably by hand. This complexity increases so rapidly, in fact, that we have found that it is largely impractical to report explicit bracket integral expressions in the open literature beyond the lowest orders even when they are organized into compact form as we have done in what follows. However, insofar as such explicit expressions can be reasonably reported in the literature, they are valuable from the point of view of having them available for general use in the existing computer codes. Further, having such explicit expressions reported in the literature, even to limited order, has a certain archival value in the field where work continues with a variety of different intermolecular potentials. Thus, in this section, we report a set of completely general and explicit expressions for the bracket integrals necessary to complete the Chapman–Enskog diffusion and thermal conductivity solutions up to order 5. Additionally, we note some basic relationships that occur between the various bracket integrals that make them more tractable to generate and manipulate in the context of the Chapman–Enskog solutions.

Of the three needed bracket integrals, the L_1 bracket integrals of Eq. (14) have the most complicated dependence upon the molecular masses of the mixture constituents. Thus, while having general expressions for these bracket integrals would have proven most useful in terms of deriving general expressions for the remaining two bracket integrals, we actually began above with the derivation of the L_{12} bracket integrals of Eq. (13) where the mass dependence is rather simple. Thus, in this section on explicit bracket integral expressions we follow the pattern established above and report explicit expressions for the L_{12} bracket integrals first. We then follow these with explicit expressions for the L_1 bracket integrals and the simple gas bracket integrals in that order. Note that for order 5 viscosity bracket integrals, one requires only $p, q \in (0,1,2,3,4)$. Thus, for the L_{12} bracket integrals up to order 5, one has:

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(0)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}(M_1M_2)^{-1} = -\frac{80}{3}\Omega_{12}^{(1)}(1) + 8\Omega_{12}^{(2)}(2), \quad (117)$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(1)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^2M_2\right)^{-1} = -\frac{280}{3}\Omega_{12}^{(1)}(1) + \frac{112}{3}\Omega_{12}^{(1)}(2) + 28\Omega_{12}^{(2)}(2) - 8\Omega_{12}^{(2)}(3), \quad (118)$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(2)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^3M_2\right)^{-1} = -210\Omega_{12}^{(1)}(1) + 168\Omega_{12}^{(1)}(2) - 24\Omega_{12}^{(1)}(3) + 63\Omega_{12}^{(2)}(2) - 36\Omega_{12}^{(2)}(3) + 4\Omega_{12}^{(2)}(4), \quad (119)$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(3)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^4M_2\right)^{-1} = -385\Omega_{12}^{(1)}(1) + 462\Omega_{12}^{(1)}(2) - 132\Omega_{12}^{(1)}(3) + \frac{88}{9}\Omega_{12}^{(1)}(4) + \frac{231}{2}\Omega_{12}^{(2)}(2) - 99\Omega_{12}^{(2)}(3) + 22\Omega_{12}^{(2)}(4) - \frac{4}{3}\Omega_{12}^{(2)}(5), \quad (120)$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(4)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^5M_2\right)^{-1} = -\frac{5005}{8}\Omega_{12}^{(1)}(1) + 1001\Omega_{12}^{(1)}(2) - 429\Omega_{12}^{(1)}(3) + \frac{572}{9}\Omega_{12}^{(1)}(4) - \frac{26}{9}\Omega_{12}^{(1)}(5) + \frac{3003}{16}\Omega_{12}^{(2)}(2) - \frac{429}{2}\Omega_{12}^{(2)}(3) + \frac{143}{2}\Omega_{12}^{(2)}(4) - \frac{26}{3}\Omega_{12}^{(2)}(5) + \frac{1}{3}\Omega_{12}^{(2)}(6), \quad (121)$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(1)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^2M_2^2\right)^{-1} = -\frac{1540}{3}\Omega_{12}^{(1)}(1) + \frac{784}{3}\Omega_{12}^{(1)}(2) - \frac{128}{3}\Omega_{12}^{(1)}(3) + \frac{602}{3}\Omega_{12}^{(2)}(2) - 56\Omega_{12}^{(2)}(3) + 8\Omega_{12}^{(2)}(4) - 16\Omega_{12}^{(3)}(3), \quad (122)$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(2)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^3M_2^2\right)^{-1} = -1575\Omega_{12}^{(1)}(1) + 1218\Omega_{12}^{(1)}(2) - 276\Omega_{12}^{(1)}(3) + 24\Omega_{12}^{(1)}(4) + \frac{1365}{2}\Omega_{12}^{(2)}(2) - 321\Omega_{12}^{(2)}(3) + 50\Omega_{12}^{(2)}(4) - 4\Omega_{12}^{(2)}(5) - 72\Omega_{12}^{(3)}(3) + 16\Omega_{12}^{(3)}(4), \quad (123)$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(3)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^4M_2^2\right)^{-1} = -\frac{7315}{2}\Omega_{12}^{(1)}(1) + 4004\Omega_{12}^{(1)}(2) - 1254\Omega_{12}^{(1)}(3) + \frac{1496}{9}\Omega_{12}^{(1)}(4) - \frac{80}{9}\Omega_{12}^{(1)}(5) + \frac{6699}{4}\Omega_{12}^{(2)}(2) - 1188\Omega_{12}^{(2)}(3) + \frac{770}{3}\Omega_{12}^{(2)}(4) - \frac{80}{3}\Omega_{12}^{(2)}(5) + \frac{4}{3}\Omega_{12}^{(2)}(6) - 198\Omega_{12}^{(3)}(3) + 88\Omega_{12}^{(3)}(4) - 8\Omega_{12}^{(3)}(5), \quad (124)$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}, S_{5/2}^{(4)}(\mathscr{C}_2^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_2\boldsymbol{\mathscr{C}}_2}\right]_{12}\left(M_1^5M_2^2\right)^{-1} = -\frac{115115}{16}\Omega_{12}^{(1)}(1) + \frac{83083}{8}\Omega_{12}^{(1)}(2) - \frac{8723}{2}\Omega_{12}^{(1)}(3) + \frac{7007}{9}\Omega_{12}^{(1)}(4) - \frac{611}{9}\Omega_{12}^{(1)}(5) + \frac{22}{9}\Omega_{12}^{(1)}(6) + \frac{109109}{32}\Omega_{12}^{(2)}(2) - \frac{52767}{16}\Omega_{12}^{(2)}(3) + \frac{11869}{12}\Omega_{12}^{(2)}(4) - \frac{2405}{18}\Omega_{12}^{(2)}(5) + \frac{59}{6}\Omega_{12}^{(2)}(6) - \frac{1}{3}\Omega_{12}^{(2)}(7) - 429\Omega_{12}^{(3)}(3) + 286\Omega_{12}^{(3)}(4) - 52\Omega_{12}^{(3)}(5) + \frac{8}{3}\Omega_{12}^{(3)}(6), \quad (125)$$

$$
\begin{aligned}
&\left[S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(2)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}\left(M_1^3M_2^3\right)^{-1} \\
&\quad = -\frac{24255}{4}\Omega^{(1)}_{12}(1) + 5670\Omega^{(1)}_{12}(2) - 1746\Omega^{(1)}_{12}(3) \\
&\qquad + 216\Omega^{(1)}_{12}(4) - 12\Omega^{(1)}_{12}(5) + \frac{25137}{8}\Omega^{(2)}_{12}(2) \\
&\qquad - 1755\Omega^{(2)}_{12}(3) + 381\Omega^{(2)}_{12}(4) - 36\Omega^{(2)}_{12}(5) + 2\Omega^{(2)}_{12}(6) \\
&\qquad - 492\Omega^{(3)}_{12}(3) + 144\Omega^{(3)}_{12}(4) - 16\Omega^{(3)}_{12}(5) + 16\Omega^{(4)}_{12}(4),
\end{aligned} \tag{126}
$$

$$
\begin{aligned}
&\left[S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(3)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}\left(M_1^4M_2^3\right)^{-1} \\
&\quad = -\frac{135135}{8}\Omega^{(1)}_{12}(1) + \frac{79695}{4}\Omega^{(1)}_{12}(2) - 7722\Omega^{(1)}_{12}(3) \\
&\qquad + 1320\Omega^{(1)}_{12}(4) - 106\Omega^{(1)}_{12}(5) + 4\Omega^{(1)}_{12}(6) + \frac{155925}{16}\Omega^{(2)}_{12}(2) \\
&\qquad - \frac{56727}{8}\Omega^{(2)}_{12}(3) + \frac{3795}{2}\Omega^{(2)}_{12}(4) - 249\Omega^{(2)}_{12}(5) + 17\Omega^{(2)}_{12}(6) \\
&\qquad - \frac{2}{3}\Omega^{(2)}_{12}(7) - 1815\Omega^{(3)}_{12}(3) + \frac{2398}{3}\Omega^{(3)}_{12}(4) - 124\Omega^{(3)}_{12}(5) \\
&\qquad + 8\Omega^{(3)}_{12}(6) + 88\Omega^{(4)}_{12}(4) - 16\Omega^{(4)}_{12}(5),
\end{aligned} \tag{127}
$$

$$
\begin{aligned}
&\left[S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}\left(M_1^5M_2^3\right)^{-1} \\
&\quad = -\frac{2477475}{64}\Omega^{(1)}_{12}(1) + \frac{909909}{16}\Omega^{(1)}_{12}(2) - \frac{433719}{16}\Omega^{(1)}_{12}(3) \\
&\qquad + \frac{11869}{2}\Omega^{(1)}_{12}(4) - \frac{2613}{4}\Omega^{(1)}_{12}(5) + 37\Omega^{(1)}_{12}(6) - \Omega^{(1)}_{12}(7) \\
&\qquad + \frac{3072069}{128}\Omega^{(2)}_{12}(2) - \frac{719433}{32}\Omega^{(2)}_{12}(3) + \frac{237237}{32}\Omega^{(2)}_{12}(4) \\
&\qquad - \frac{4901}{4}\Omega^{(2)}_{12}(5) + \frac{891}{8}\Omega^{(2)}_{12}(6) - \frac{35}{6}\Omega^{(2)}_{12}(7) + \frac{1}{6}\Omega^{(2)}_{12}(8) \\
&\qquad - \frac{9867}{2}\Omega^{(3)}_{12}(3) + \frac{9152}{3}\Omega^{(3)}_{12}(4) - \frac{1924}{3}\Omega^{(3)}_{12}(5) \\
&\qquad + 64\Omega^{(3)}_{12}(6) - \frac{8}{3}\Omega^{(3)}_{12}(7) + 286\Omega^{(4)}_{12}(4) - 104\Omega^{(4)}_{12}(5) \\
&\qquad + 8\Omega^{(4)}_{12}(6)\ ,
\end{aligned} \tag{128}
$$

$$
\begin{aligned}
&\left[S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(3)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}\left(M_1^4M_2^4\right)^{-1} \\
&\quad = -\frac{875875}{16}\Omega^{(1)}_{12}(1) + \frac{297297}{4}\Omega^{(1)}_{12}(2) - \frac{69267}{2}\Omega^{(1)}_{12}(3) \\
&\qquad + \frac{22264}{3}\Omega^{(1)}_{12}(4) - \frac{2425}{3}\Omega^{(1)}_{12}(5) + 44\Omega^{(1)}_{12}(6) - \frac{32}{27}\Omega^{(1)}_{12}(7) \\
&\qquad + \frac{1168167}{32}\Omega^{(2)}_{12}(2) - \frac{245025}{8}\Omega^{(2)}_{12}(3) + \frac{236005}{24}\Omega^{(2)}_{12}(4) \\
&\qquad - 1573\Omega^{(2)}_{12}(5) + \frac{859}{6}\Omega^{(2)}_{12}(6) - \frac{22}{3}\Omega^{(2)}_{12}(7) + \frac{2}{9}\Omega^{(2)}_{12}(8) \\
&\qquad - \frac{33737}{4}\Omega^{(3)}_{12}(3) + \frac{13310}{3}\Omega^{(3)}_{12}(4) - \frac{8290}{9}\Omega^{(3)}_{12}(5) \\
&\qquad + 88\Omega^{(3)}_{12}(6) - 4\Omega^{(3)}_{12}(7) + \frac{5852}{9}\Omega^{(4)}_{12}(4) - 176\Omega^{(4)}_{12}(5) \\
&\qquad + 16\Omega^{(4)}_{12}(6) - \frac{32}{3}\Omega^{(5)}_{12}(5),
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
&\left[S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}\left(M_1^5M_2^4\right)^{-1} \\
&\quad = -\frac{18293275}{128}\Omega^{(1)}_{12}(1) + \frac{14559545}{64}\Omega^{(1)}_{12}(2) - \frac{3990129}{32}\Omega^{(1)}_{12}(3) \\
&\qquad + \frac{1559701}{48}\Omega^{(1)}_{12}(4) - \frac{107939}{24}\Omega^{(1)}_{12}(5) + \frac{4057}{12}\Omega^{(1)}_{12}(6) \\
&\qquad - \frac{713}{54}\Omega^{(1)}_{12}(7) + \frac{7}{27}\Omega^{(1)}_{12}(8) + \frac{26951925}{256}\Omega^{(2)}_{12}(2) \\
&\qquad - \frac{13505349}{128}\Omega^{(2)}_{12}(3) + \frac{7661797}{192}\Omega^{(2)}_{12}(4) \\
&\qquad - \frac{738569}{96}\Omega^{(2)}_{12}(5) + \frac{41005}{48}\Omega^{(2)}_{12}(6) - \frac{4175}{72}\Omega^{(2)}_{12}(7) \\
&\qquad + \frac{85}{36}\Omega^{(2)}_{12}(8) - \frac{1}{18}\Omega^{(2)}_{12}(9) - \frac{221507}{8}\Omega^{(3)}_{12}(3) \\
&\qquad + \frac{655655}{36}\Omega^{(3)}_{12}(4) - \frac{41873}{9}\Omega^{(3)}_{12}(5) + \frac{5386}{9}\Omega^{(3)}_{12}(6) \\
&\qquad - \frac{122}{3}\Omega^{(3)}_{12}(7) + \frac{4}{3}\Omega^{(3)}_{12}(8) + \frac{23881}{9}\Omega^{(4)}_{12}(4) - \frac{9490}{9}\Omega^{(4)}_{12}(5) \\
&\qquad + 148\Omega^{(4)}_{12}(6) - 8\Omega^{(4)}_{12}(7) - \frac{208}{3}\Omega^{(5)}_{12}(5) + \frac{32}{3}\Omega^{(5)}_{12}(6),
\end{aligned} \tag{130}
$$

$$
\begin{aligned}
&\left[S^{(4)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_2^2)\boldsymbol{\mathscr{C}}_2\mathring{\boldsymbol{\mathscr{C}}}_2\right]_{12}\left(M_1^5M_2^5\right)^{-1} \\
&\quad = -\frac{428402975}{1024}\Omega^{(1)}_{12}(1) + \frac{47562515}{64}\Omega^{(1)}_{12}(2) \\
&\qquad - \frac{29713255}{64}\Omega^{(1)}_{12}(3) + \frac{6779773}{48}\Omega^{(1)}_{12}(4) - \frac{758173}{32}\Omega^{(1)}_{12}(5) \\
&\qquad + \frac{27287}{12}\Omega^{(1)}_{12}(6) - \frac{13213}{108}\Omega^{(1)}_{12}(7) + \frac{91}{27}\Omega^{(1)}_{12}(8) - \frac{5}{108}\Omega^{(1)}_{12}(9) \\
&\qquad + \frac{711736025}{2048}\Omega^{(2)}_{12}(2) - \frac{50053575}{128}\Omega^{(2)}_{12}(3) \\
&\qquad + \frac{64969619}{384}\Omega^{(2)}_{12}(4) - \frac{3630289}{96}\Omega^{(2)}_{12}(5) + \frac{953681}{192}\Omega^{(2)}_{12}(6) \\
&\qquad - \frac{29393}{72}\Omega^{(2)}_{12}(7) + \frac{4699}{216}\Omega^{(2)}_{12}(8) - \frac{13}{18}\Omega^{(2)}_{12}(9) + \frac{1}{72}\Omega^{(2)}_{12}(10) \\
&\qquad - \frac{1705275}{16}\Omega^{(3)}_{12}(3) + \frac{2879591}{36}\Omega^{(3)}_{12}(4) - \frac{289133}{12}\Omega^{(3)}_{12}(5) \\
&\qquad + \frac{33982}{9}\Omega^{(3)}_{12}(6) - \frac{3061}{9}\Omega^{(3)}_{12}(7) + \frac{52}{3}\Omega^{(3)}_{12}(8) - \frac{4}{9}\Omega^{(3)}_{12}(9) \\
&\qquad + \frac{470041}{36}\Omega^{(4)}_{12}(4) - \frac{56446}{9}\Omega^{(4)}_{12}(5) + \frac{10682}{9}\Omega^{(4)}_{12}(6) \\
&\qquad - 104\Omega^{(4)}_{12}(7) + 4\Omega^{(4)}_{12}(8) - \frac{5096}{9}\Omega^{(5)}_{12}(5) + \frac{416}{3}\Omega^{(5)}_{12}(6) \\
&\qquad - \frac{32}{3}\Omega^{(5)}_{12}(7) + \frac{16}{3}\Omega^{(6)}_{12}(6).
\end{aligned} \tag{131}
$$

Next, we consider the L_1 bracket integrals of Eq. (14). For each (p,q), these may be used to generate the corresponding L_{12} expressions in Eqs. (117)–(131) above. In general, the expressions for the L_1 bracket integrals consist of a series of terms each of which is associated with a specific omega integral. Each of these omega integral terms also contains a sign, a constant factor, and some function of M_1 and M_2. The conversion process from the L_1 expressions to the L_{12} expressions is quite straightforward as the magnitude of the numerical coefficients associated with each omega integral term are the same from the L_1 expressions to the L_{12} expressions. The differences are in the distributions of the constituent masses and the signs of the terms. Since the same omega integrals must occur in both the L_1 expressions and the L_{12} expressions, the total number of omega integral terms must be the same in each and there is, in effect, a one-to-one correspondence between terms. The sign difference between corresponding terms is dependent only upon ℓ and the appropriate sign transformation is to multiply each term in the L_1 expressions that follow by a factor of $(-1)^{\ell}$. The distribution of constituent masses is much simpler in the L_{12} expressions than in the L_1 expressions. In the L_{12} expressions given above in Eqs. (117)–(131), the distribution of the constituent masses is exactly the same in every omega integral term for a given (p, q) and amounts to nothing more than a common factor of $(M_1^{(q+1)}M_2^{(p+1)})$ in each expression. Thus, overall, the appropriate transformation from the L_1 expressions that follow to the L_{12} expressions in Eqs. (117)–(131) is to first set $M_1 = M_2 = 1$ in each term in the L_1 expressions, second to multiply each expression with the appropriate common factor of $(M_1^{(q+1)}M_2^{(p+1)})$, and third to multiply each term by a factor of $(-1)^{\ell}$. When the above prescription is followed, the expressions below yield the corresponding L_{12} bracket integrals reported in Eqs. (117)–(131). The explicit expressions for the L_1 bracket integrals up to order 5 are:

$$\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(0)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} = \frac{80}{3}(M_1M_2)\Omega^{(1)}_{12}(1) + 8\left(M_2^2\right)\Omega^{(2)}_{12}(2), \tag{132}$$

$$\begin{aligned}\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(1)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= \frac{280}{3}\left(M_1M_2^2\right)\Omega^{(1)}_{12}(1) - \frac{112}{3}\left(M_1M_2^2\right)\Omega^{(1)}_{12}(2) \\ &\quad + 28\left(M_2^3\right)\Omega^{(2)}_{12}(2) - 8\left(M_2^3\right)\Omega^{(2)}_{12}(3),\end{aligned} \tag{133}$$

$$\begin{aligned}\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= 210\left(M_1M_2^3\right)\Omega^{(1)}_{12}(1) - 168\left(M_1M_2^3\right)\Omega^{(1)}_{12}(2) \\ &\quad + 24\left(M_1M_2^3\right)\Omega^{(1)}_{12}(3) + 63\left(M_2^4\right)\Omega^{(2)}_{12}(2) \\ &\quad - 36\left(M_2^4\right)\Omega^{(2)}_{12}(3) + 4\left(M_2^4\right)\Omega^{(2)}_{12}(4),\end{aligned} \tag{134}$$

$$\begin{aligned}\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= 385\left(M_1M_2^4\right)\Omega^{(1)}_{12}(1) - 462\left(M_1M_2^4\right)\Omega^{(1)}_{12}(2) \\ &\quad + 132\left(M_1M_2^4\right)\Omega^{(1)}_{12}(3) - \frac{88}{9}\left(M_1M_2^4\right)\Omega^{(1)}_{12}(4) \\ &\quad + \frac{231}{2}\left(M_2^5\right)\Omega^{(2)}_{12}(2) - 99\left(M_2^5\right)\Omega^{(2)}_{12}(3) \\ &\quad + 22\left(M_2^5\right)\Omega^{(2)}_{12}(4) - \frac{4}{3}\left(M_2^5\right)\Omega^{(2)}_{12}(5),\end{aligned} \tag{135}$$

$$\begin{aligned}\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= \frac{5005}{8}\left(M_1M_2^5\right)\Omega^{(1)}_{12}(1) - 1001\left(M_1M_2^5\right)\Omega^{(1)}_{12}(2) \\ &\quad + 429\left(M_1M_2^5\right)\Omega^{(1)}_{12}(3) - \frac{572}{9}\left(M_1M_2^5\right)\Omega^{(1)}_{12}(4) \\ &\quad + \frac{26}{9}\left(M_1M_2^5\right)\Omega^{(1)}_{12}(5) + \frac{3003}{16}\left(M_2^6\right)\Omega^{(2)}_{12}(2) \\ &\quad - \frac{429}{2}\left(M_2^6\right)\Omega^{(2)}_{12}(3) + \frac{143}{2}\left(M_2^6\right)\Omega^{(2)}_{12}(4) \\ &\quad - \frac{26}{3}\left(M_2^6\right)\Omega^{(2)}_{12}(5) + \frac{1}{3}\left(M_2^6\right)\Omega^{(2)}_{12}(6)\ ,\end{aligned} \tag{136}$$

$$\begin{aligned}\left[S^{(1)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(1)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= \frac{1540}{3}\left(\frac{4}{11}M_1^3M_2 + \frac{7}{11}M_1M_2^3\right)\Omega^{(1)}_{12}(1) \\ &\quad - \frac{784}{3}\left(M_1M_2^3\right)\Omega^{(1)}_{12}(2) + \frac{128}{3}\left(M_1M_2^3\right)\Omega^{(1)}_{12}(3) \\ &\quad + \frac{602}{3}\left(\frac{22}{43}M_1^2M_2^2 + \frac{21}{43}M_2^4\right)\Omega^{(2)}_{12}(2) - 56\left(M_2^4\right)\Omega^{(2)}_{12}(3) \\ &\quad + 8\left(M_2^4\right)\Omega^{(2)}_{12}(4) + 16\left(M_1M_2^3\right)\Omega^{(3)}_{12}(3),\end{aligned} \tag{137}$$

$$\begin{aligned}\left[S^{(1)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= 1575\left(\frac{8}{15}M_1^3M_2^2 + \frac{7}{15}M_1M_2^4\right)\Omega^{(1)}_{12}(1) \\ &\quad - 1218\left(\frac{8}{29}M_1^3M_2^2 + \frac{21}{29}M_1M_2^4\right)\Omega^{(1)}_{12}(2) \\ &\quad + 276\left(M_1M_2^4\right)\Omega^{(1)}_{12}(3) - 24\left(M_1M_2^4\right)\Omega^{(1)}_{12}(4) \\ &\quad + \frac{1365}{2}\left(\frac{44}{65}M_1^2M_2^3 + \frac{21}{65}M_2^5\right)\Omega^{(2)}_{12}(2) \\ &\quad - 321\left(\frac{44}{107}M_1^2M_2^3 + \frac{63}{107}M_2^5\right)\Omega^{(2)}_{12}(3) \\ &\quad + 50\left(M_2^5\right)\Omega^{(2)}_{12}(4) - 4\left(M_2^5\right)\Omega^{(2)}_{12}(5) \\ &\quad + 72\left(M_1M_2^4\right)\Omega^{(3)}_{12}(3) - 16\left(M_1M_2^4\right)\Omega^{(3)}_{12}(4),\end{aligned} \tag{138}$$

$$\begin{aligned}\left[S^{(1)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= \frac{7315}{2}\left(\frac{12}{19}M_1^3M_2^3 + \frac{7}{19}M_1M_2^5\right)\Omega^{(1)}_{12}(1) \\ &\quad - 4004\left(\frac{6}{13}M_1^3M_2^3 + \frac{7}{13}M_1M_2^5\right)\Omega^{(1)}_{12}(2) \\ &\quad + 1254\left(\frac{4}{19}M_1^3M_2^3 + \frac{15}{19}M_1M_2^5\right)\Omega^{(1)}_{12}(3) \\ &\quad - \frac{1496}{9}\left(M_1M_2^5\right)\Omega^{(1)}_{12}(4) + \frac{80}{9}\left(M_1M_2^5\right)\Omega^{(1)}_{12}(5) \\ &\quad + \frac{6699}{4}\left(\frac{22}{29}M_1^2M_2^4 + \frac{7}{29}M_2^6\right)\Omega^{(2)}_{12}(2) \\ &\quad - 1188\left(\frac{11}{18}M_1^2M_2^4 + \frac{7}{18}M_2^6\right)\Omega^{(2)}_{12}(3) \\ &\quad + \frac{770}{3}\left(\frac{11}{35}M_1^2M_2^4 + \frac{24}{35}M_2^6\right)\Omega^{(2)}_{12}(4) - \frac{80}{3}\left(M_2^6\right)\Omega^{(2)}_{12}(5) \\ &\quad + \frac{4}{3}\left(M_2^6\right)\Omega^{(2)}_{12}(6) + 198\left(M_1M_2^5\right)\Omega^{(3)}_{12}(3) \\ &\quad - 88\left(M_1M_2^5\right)\Omega^{(3)}_{12}(4) + 8\left(M_1M_2^5\right)\Omega^{(3)}_{12}(5),\end{aligned} \tag{139}$$

$$\begin{aligned}\left[S^{(1)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12} &= \frac{115115}{16}\left(\frac{16}{23}M_1^3M_2^4 + \frac{7}{23}M_1M_2^6\right)\Omega^{(1)}_{12}(1) \\ &\quad - \frac{83083}{8}\left(\frac{48}{83}M_1^3M_2^4 + \frac{35}{83}M_1M_2^6\right)\Omega^{(1)}_{12}(2) \\ &\quad + \frac{8723}{2}\left(\frac{24}{61}M_1^3M_2^4 + \frac{37}{61}M_1M_2^6\right)\Omega^{(1)}_{12}(3) \\ &\quad - \frac{7007}{9}\left(\frac{8}{49}M_1^3M_2^4 + \frac{41}{49}M_1M_2^6\right)\Omega^{(1)}_{12}(4) + \frac{611}{9}\left(M_1M_2^6\right)\Omega^{(1)}_{12}(5) \\ &\quad - \frac{22}{9}\left(M_1M_2^6\right)\Omega^{(1)}_{12}(6) + \frac{109109}{32}\left(\frac{88}{109}M_1^2M_2^5 + \frac{21}{109}M_2^7\right)\Omega^{(2)}_{12}(2) \\ &\quad - \frac{52767}{16}\left(\frac{88}{123}M_1^2M_2^5 + \frac{35}{123}M_2^7\right)\Omega^{(2)}_{12}(3) \\ &\quad + \frac{11869}{12}\left(\frac{44}{83}M_1^2M_2^5 + \frac{39}{83}M_2^7\right)\Omega^{(2)}_{12}(4) \\ &\quad - \frac{2405}{18}\left(\frac{44}{185}M_1^2M_2^5 + \frac{141}{185}M_2^7\right)\Omega^{(2)}_{12}(5) + \frac{59}{6}\left(M_2^7\right)\Omega^{(2)}_{12}(6) \\ &\quad - \frac{1}{3}\left(M_2^7\right)\Omega^{(2)}_{12}(7) + 429\left(M_1M_2^6\right)\Omega^{(3)}_{12}(3) \\ &\quad - 286\left(M_1M_2^6\right)\Omega^{(3)}_{12}(4) + 52\left(M_1M_2^6\right)\Omega^{(3)}_{12}(5) \\ &\quad - \frac{8}{3}\left(M_1M_2^6\right)\Omega^{(3)}_{12}(6),\end{aligned} \tag{140}$$

$$\left[S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}$$

$$\begin{aligned}
&= \frac{24255}{4}\left(\frac{8}{77}M_1^5M_2+\frac{48}{77}M_1^3M_2^3+\frac{21}{77}M_1M_2^5\right)\Omega^{(1)}_{12}(1)-5670\left(\frac{8}{15}M_1^3M_2^3+\frac{7}{15}M_1M_2^5\right)\Omega^{(1)}_{12}(2)+1746\left(\frac{28}{97}M_1^3M_2^3+\frac{69}{97}M_1M_2^5\right)\Omega^{(1)}_{12}(3)\\
&\quad -216\left(M_1M_2^5\right)\Omega^{(1)}_{12}(4)+12\left(M_1M_2^5\right)\Omega^{(1)}_{12}(5)+\frac{25137}{8}\left(\frac{24}{133}M_1^4M_2^2+\frac{88}{133}M_1^2M_2^4+\frac{21}{133}M_2^6\right)\Omega^{(2)}_{12}(2)\\
&\quad -1755\left(\frac{44}{65}M_1^2M_2^4+\frac{21}{65}M_2^6\right)\Omega^{(2)}_{12}(3)+381\left(\frac{52}{127}M_1^2M_2^4+\frac{75}{127}M_2^6\right)\Omega^{(2)}_{12}(4)-36\left(M_2^6\right)\Omega^{(2)}_{12}(5)+2\left(M_2^6\right)\Omega^{(2)}_{12}(6)\\
&\quad +492\left(\frac{14}{41}M_1^3M_2^3+\frac{27}{41}M_1M_2^5\right)\Omega^{(3)}_{12}(3)-144\left(M_1M_2^5\right)\Omega^{(3)}_{12}(4)+16\left(M_1M_2^5\right)\Omega^{(3)}_{12}(5)+16\left(M_1^2M_2^4\right)\Omega^{(4)}_{12}(4),
\end{aligned} \tag{141}$$

$$\left[S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}$$

$$\begin{aligned}
&= \frac{135135}{8}\left(\frac{8}{39}M_1^5M_2^2+\frac{24}{39}M_1^3M_2^4+\frac{7}{39}M_1M_2^6\right)\Omega^{(1)}_{12}(1)-\frac{79695}{4}\left(\frac{8}{115}M_1^5M_2^2+\frac{72}{115}M_1^3M_2^4+\frac{35}{115}M_1M_2^6\right)\Omega^{(1)}_{12}(2)\\
&\quad +7722\left(\frac{20}{39}M_1^3M_2^4+\frac{19}{39}M_1M_2^6\right)\Omega^{(1)}_{12}(3)-1320\left(\frac{4}{15}M_1^3M_2^4+\frac{11}{15}M_1M_2^6\right)\Omega^{(1)}_{12}(4)+106\left(M_1M_2^6\right)\Omega^{(1)}_{12}(5)-4\left(M_1M_2^6\right)\Omega^{(1)}_{12}(6)\\
&\quad +\frac{155925}{16}\left(\frac{24}{75}M_1^4M_2^3+\frac{44}{75}M_1^2M_2^5+\frac{7}{75}M_2^7\right)\Omega^{(2)}_{12}(2)-\frac{56727}{8}\left(\frac{24}{191}M_1^4M_2^3+\frac{132}{191}M_1^2M_2^5+\frac{35}{191}M_2^7\right)\Omega^{(2)}_{12}(3)\\
&\quad +\frac{3795}{2}\left(\frac{74}{115}M_1^2M_2^5+\frac{41}{115}M_2^7\right)\Omega^{(2)}_{12}(4)-249\left(\frac{30}{83}M_1^2M_2^5+\frac{53}{83}M_2^7\right)\Omega^{(2)}_{12}(5)+17\left(M_2^7\right)\Omega^{(2)}_{12}(6)-\frac{2}{3}\left(M_2^7\right)\Omega^{(2)}_{12}(7)\\
&\quad +1815\left(\frac{924}{1815}M_1^3M_2^4+\frac{891}{1815}M_1M_2^6\right)\Omega^{(3)}_{12}(3)-\frac{2398}{3}\left(\frac{28}{109}M_1^3M_2^4+\frac{81}{109}M_1M_2^6\right)\Omega^{(3)}_{12}(4)\\
&\quad +124\left(M_1M_2^6\right)\Omega^{(3)}_{12}(5)-8\left(M_1M_2^6\right)\Omega^{(3)}_{12}(6)+88\left(M_1^2M_2^5\right)\Omega^{(4)}_{12}(4)-16\left(M_1^2M_2^5\right)\Omega^{(4)}_{12}(5),
\end{aligned} \tag{142}$$

$$\left[S^{(2)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}$$

$$\begin{aligned}
&= \frac{2477475}{64}\left(\frac{16}{55}M_1^5M_2^3+\frac{32}{55}M_1^3M_2^5+\frac{7}{55}M_1M_2^7\right)\Omega^{(1)}_{12}(1)-\frac{909909}{16}\left(\frac{16}{101}M_1^5M_2^3+\frac{64}{101}M_1^3M_2^5+\frac{21}{101}M_1M_2^7\right)\Omega^{(1)}_{12}(2)\\
&\quad +\frac{433719}{16}\left(\frac{16}{337}M_1^5M_2^3+\frac{208}{337}M_1^3M_2^5+\frac{113}{337}M_1M_2^7\right)\Omega^{(1)}_{12}(3)-\frac{11869}{2}\left(\frac{40}{83}M_1^3M_2^5+\frac{43}{83}M_1M_2^7\right)\Omega^{(1)}_{12}(4)\\
&\quad +\frac{2613}{4}\left(\frac{16}{67}M_1^3M_2^5+\frac{51}{67}M_1M_2^7\right)\Omega^{(1)}_{12}(5)-37\left(M_1M_2^7\right)\Omega^{(1)}_{12}(6)+\left(M_1M_2^7\right)\Omega^{(1)}_{12}(7)\\
&\quad +\frac{3072069}{128}\left(\frac{144}{341}M_1^4M_2^4+\frac{176}{341}M_1^2M_2^6+\frac{21}{341}M_2^8\right)\Omega^{(2)}_{12}(2)-\frac{719433}{32}\left(\frac{144}{559}M_1^4M_2^4+\frac{352}{559}M_1^2M_2^6+\frac{63}{559}M_2^8\right)\Omega^{(2)}_{12}(3)\\
&\quad +\frac{237237}{32}\left(\frac{48}{553}M_1^4M_2^4+\frac{384}{553}M_1^2M_2^6+\frac{121}{553}M_2^8\right)\Omega^{(2)}_{12}(4)-\frac{4901}{4}\left(\frac{224}{377}M_1^2M_2^6+\frac{153}{377}M_2^8\right)\Omega^{(2)}_{12}(5)\\
&\quad +\frac{891}{8}\left(\frac{272}{891}M_1^2M_2^6+\frac{619}{891}M_2^8\right)\Omega^{(2)}_{12}(6)-\frac{35}{6}\left(M_2^8\right)\Omega^{(2)}_{12}(7)+\frac{1}{6}\left(M_2^8\right)\Omega^{(2)}_{12}(8)+\frac{9867}{2}\left(\frac{14}{23}M_1^3M_2^5+\frac{9}{23}M_1M_2^7\right)\Omega^{(3)}_{12}(3)\\
&\quad -\frac{9152}{3}\left(\frac{7}{16}M_1^3M_2^5+\frac{9}{16}M_1M_2^7\right)\Omega^{(3)}_{12}(4)+\frac{1924}{3}\left(\frac{7}{37}M_1^3M_2^5+\frac{30}{37}M_1M_2^7\right)\Omega^{(3)}_{12}(5)-64\left(M_1M_2^7\right)\Omega^{(3)}_{12}(6)\\
&\quad +\frac{8}{3}\left(M_1M_2^7\right)\Omega^{(3)}_{12}(7)+286\left(M_1^2M_2^6\right)\Omega^{(4)}_{12}(4)-104\left(M_1^2M_2^6\right)\Omega^{(4)}_{12}(5)+8\left(M_1^2M_2^6\right)\Omega^{(4)}_{12}(6)\ ,
\end{aligned} \tag{143}$$

$$
\begin{aligned}
&\left[S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}\\
&\quad= \frac{875875}{16}\left(\frac{64}{2275}M_1^7M_2+\frac{792}{2275}M_1^5M_2^3+\frac{1188}{2275}M_1^3M_2^5+\frac{231}{2275}M_1M_2^7\right)\Omega_{12}^{(1)}(1)\\
&\qquad-\frac{297297}{4}\left(\frac{8}{39}M_1^5M_2^3+\frac{24}{39}M_1^3M_2^5+\frac{7}{39}M_1M_2^7\right)\Omega_{12}^{(1)}(2)+\frac{69267}{2}\left(\frac{152}{2099}M_1^5M_2^3+\frac{1320}{2099}M_1^3M_2^5+\frac{627}{2099}M_1M_2^7\right)\Omega_{12}^{(1)}(3)\\
&\qquad-\frac{22264}{3}\left(\frac{12}{23}M_1^3M_2^5+\frac{11}{23}M_1M_2^7\right)\Omega_{12}^{(1)}(4)+\frac{2425}{3}\left(\frac{676}{2425}M_1^3M_2^5+\frac{1749}{2425}M_1M_2^7\right)\Omega_{12}^{(1)}(5)-44\left(M_1M_2^7\right)\Omega_{12}^{(1)}(6)\\
&\qquad+\frac{32}{27}\left(M_1M_2^7\right)\Omega_{12}^{(1)}(7)+\frac{1168167}{32}\left(\frac{272}{5057}M_1^6M_2^2+\frac{2376}{5057}M_1^4M_2^4+\frac{2178}{5057}M_1^2M_2^6+\frac{231}{5057}M_2^8\right)\Omega_{12}^{(2)}(2)\\
&\qquad-\frac{245025}{8}\left(\frac{24}{75}M_1^4M_2^4+\frac{44}{75}M_1^2M_2^6+\frac{7}{75}M_2^8\right)\Omega_{12}^{(2)}(3)+\frac{236005}{24}\left(\frac{2744}{21455}M_1^4M_2^4+\frac{14652}{21455}M_1^2M_2^6+\frac{4059}{21455}M_2^8\right)\Omega_{12}^{(2)}(4)\\
&\qquad-1573\left(\frac{90}{143}M_1^2M_2^6+\frac{53}{143}M_2^8\right)\Omega_{12}^{(2)}(5)+\frac{859}{6}\left(\frac{298}{859}M_1^2M_2^6+\frac{561}{859}M_2^8\right)\Omega_{12}^{(2)}(6)-\frac{22}{3}\left(M_2^8\right)\Omega_{12}^{(2)}(7)+\frac{2}{9}\left(M_2^8\right)\Omega_{12}^{(2)}(8)\\
&\qquad+\frac{33737}{4}\left(\frac{328}{3067}M_1^5M_2^3+\frac{1848}{3067}M_1^3M_2^5+\frac{891}{3067}M_1M_2^7\right)\Omega_{12}^{(3)}(3)-\frac{13310}{3}\left(\frac{28}{55}M_1^3M_2^5+\frac{27}{55}M_1M_2^7\right)\Omega_{12}^{(3)}(4)\\
&\qquad+\frac{8290}{9}\left(\frac{1076}{4145}M_1^3M_2^5+\frac{3069}{4145}M_1M_2^7\right)\Omega_{12}^{(3)}(5)-88\left(M_1M_2^7\right)\Omega_{12}^{(3)}(6)+4\left(M_1M_2^7\right)\Omega_{12}^{(3)}(7)\\
&\qquad+\frac{5852}{9}\left(\frac{34}{133}M_1^4M_2^4+\frac{99}{133}M_1^2M_2^6\right)\Omega_{12}^{(4)}(4)-176\left(M_1^2M_2^6\right)\Omega_{12}^{(4)}(5)+16\left(M_1^2M_2^6\right)\Omega_{12}^{(4)}(6)+\frac{32}{3}\left(M_1^3M_2^5\right)\Omega_{12}^{(5)}(5),
\end{aligned}
\tag{144}
$$

$$
\begin{aligned}
&\left[S^{(3)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1, S^{(4)}_{5/2}(\mathscr{C}_1^2)\boldsymbol{\mathscr{C}}_1\mathring{\boldsymbol{\mathscr{C}}}_1\right]_{12}\\
&\quad= \frac{18293275}{128}\left(\frac{256}{3655}M_1^7M_2^2+\frac{1584}{3655}M_1^5M_2^4+\frac{1584}{3655}M_1^3M_2^6+\frac{231}{3655}M_1M_2^8\right)\Omega_{12}^{(1)}(1)\\
&\qquad-\frac{14559545}{64}\left(\frac{256}{14545}M_1^7M_2^2+\frac{4752}{14545}M_1^5M_2^4+\frac{7920}{14545}M_1^3M_2^6+\frac{1617}{14545}M_1M_2^8\right)\Omega_{12}^{(1)}(2)\\
&\qquad+\frac{3990129}{32}\left(\frac{1744}{9301}M_1^5M_2^4+\frac{5808}{9301}M_1^3M_2^6+\frac{1749}{9301}M_1M_2^8\right)\Omega_{12}^{(1)}(3)\\
&\qquad-\frac{1559701}{48}\left(\frac{688}{10907}M_1^5M_2^4+\frac{6864}{10907}M_1^3M_2^6+\frac{3355}{10907}M_1M_2^8\right)\Omega_{12}^{(1)}(4)+\frac{107939}{24}\left(\frac{4288}{8303}M_1^3M_2^6+\frac{4015}{8303}M_1M_2^8\right)\Omega_{12}^{(1)}(5)\\
&\qquad-\frac{4057}{12}\left(\frac{1120}{4057}M_1^3M_2^6+\frac{2937}{4057}M_1M_2^8\right)\Omega_{12}^{(1)}(6)+\frac{713}{54}\left(M_1M_2^8\right)\Omega_{12}^{(1)}(7)-\frac{7}{27}\left(M_1M_2^8\right)\Omega_{12}^{(1)}(8)\\
&\qquad+\frac{26951925}{256}\left(\frac{1088}{8975}M_1^6M_2^3+\frac{4752}{8975}M_1^4M_2^5+\frac{2904}{8975}M_1^2M_2^7+\frac{231}{8975}M_2^9\right)\Omega_{12}^{(2)}(2)\\
&\qquad-\frac{13505349}{128}\left(\frac{1088}{31481}M_1^6M_2^3+\frac{14256}{31481}M_1^4M_2^5+\frac{14520}{31481}M_1^2M_2^7+\frac{1617}{31481}M_2^9\right)\Omega_{12}^{(2)}(3)\\
&\qquad+\frac{7661797}{192}\left(\frac{15728}{53579}M_1^4M_2^5+\frac{32208}{53579}M_1^2M_2^7+\frac{5643}{53579}M_2^9\right)\Omega_{12}^{(2)}(4)-\frac{738569}{96}\left(\frac{6224}{56813}M_1^4M_2^5+\frac{38544}{56813}M_1^2M_2^7+\frac{12045}{56813}M_2^9\right)\Omega_{12}^{(2)}(5)\\
&\qquad+\frac{41005}{48}\left(\frac{24472}{41005}M_1^2M_2^7+\frac{16533}{41005}M_2^9\right)\Omega_{12}^{(2)}(6)-\frac{4175}{72}\left(\frac{1304}{4175}M_1^2M_2^7+\frac{2871}{4175}M_2^9\right)\Omega_{12}^{(2)}(7)+\frac{85}{36}\left(M_2^9\right)\Omega_{12}^{(2)}(8)-\frac{1}{18}\left(M_2^9\right)\Omega_{12}^{(2)}(9)\\
&\qquad+\frac{221507}{8}\left(\frac{328}{1549}M_1^5M_2^4+\frac{924}{1549}M_1^3M_2^6+\frac{297}{1549}M_1M_2^8\right)\Omega_{12}^{(3)}(3)-\frac{655655}{36}\left(\frac{328}{4585}M_1^5M_2^4+\frac{2772}{4585}M_1^3M_2^6+\frac{1485}{4585}M_1M_2^8\right)\Omega_{12}^{(3)}(4)\\
&\qquad+\frac{41873}{9}\left(\frac{1538}{3221}M_1^3M_2^6+\frac{1683}{3221}M_1M_2^8\right)\Omega_{12}^{(3)}(5)-\frac{5386}{9}\left(\frac{614}{2693}M_1^3M_2^6+\frac{2079}{2693}M_1M_2^8\right)\Omega_{12}^{(3)}(6)+\frac{122}{3}\left(M_1M_2^8\right)\Omega_{12}^{(3)}(7)\\
&\qquad-\frac{4}{3}\left(M_1M_2^8\right)\Omega_{12}^{(3)}(8)+\frac{23881}{9}\left(\frac{68}{167}M_1^4M_2^5+\frac{99}{167}M_1^2M_2^7\right)\Omega_{12}^{(4)}(4)-\frac{9490}{9}\left(\frac{68}{365}M_1^4M_2^5+\frac{297}{365}M_1^2M_2^7\right)\Omega_{12}^{(4)}(5)\\
&\qquad+148\left(M_1^2M_2^7\right)\Omega_{12}^{(4)}(6)-8\left(M_1^2M_2^7\right)\Omega_{12}^{(4)}(7)+\frac{208}{3}\left(M_1^3M_2^6\right)\Omega_{12}^{(5)}(5)-\frac{32}{3}\left(M_1^3M_2^6\right)\Omega_{12}^{(5)}(6),
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
&\left[S^{(4)}_{5/2}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},S^{(4)}_{5/2}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right]_{12}\\
&=\frac{428402975}{1024}\left(\frac{640}{85595}M_1^9M_2+\frac{13312}{85595}M_1^7M_2^3+\frac{41184}{85595}M_1^5M_2^5+\frac{27456}{85595}M_1^3M_2^7+\frac{3003}{85595}M_1M_2^9\right)\Omega_{12}^{(1)}(1)\\
&\quad-\frac{47562515}{64}\left(\frac{256}{3655}M_1^7M_2^3+\frac{1584}{3655}M_1^5M_2^5+\frac{1584}{3655}M_1^3M_2^7+\frac{231}{3655}M_1M_2^9\right)\Omega_{12}^{(1)}(2)\\
&\quad+\frac{29713255}{64}\left(\frac{3776}{207785}M_1^7M_2^3+\frac{68016}{207785}M_1^5M_2^5+\frac{113256}{207785}M_1^3M_2^7+\frac{22737}{207785}M_1M_2^9\right)\Omega_{12}^{(1)}(3)\\
&\quad-\frac{6779773}{48}\left(\frac{688}{3647}M_1^5M_2^5+\frac{2288}{3647}M_1^3M_2^7+\frac{671}{3647}M_1M_2^9\right)\Omega_{12}^{(1)}(4)+\frac{758173}{32}\left(\frac{11280}{174963}M_1^5M_2^5+\frac{111488}{174963}M_1^3M_2^7+\frac{52195}{174963}M_1M_2^9\right)\Omega_{12}^{(1)}(5)\\
&\quad-\frac{27287}{12}\left(\frac{1120}{2099}M_1^3M_2^7+\frac{979}{2099}M_1M_2^9\right)\Omega_{12}^{(1)}(6)+\frac{13213}{108}\left(\frac{3944}{13213}M_1^3M_2^7+\frac{9269}{13213}M_1M_2^9\right)\Omega_{12}^{(1)}(7)-\frac{91}{27}\left(M_1M_2^9\right)\Omega_{12}^{(1)}(8)\\
&\quad+\frac{5}{108}\left(M_1M_2^9\right)\Omega_{12}^{(1)}(9)+\frac{711736025}{2048}\left(\frac{10624}{711025}M_1^8M_2^2+\frac{169728}{711025}M_1^6M_2^4+\frac{370656}{711025}M_1^4M_2^6+\frac{151008}{711025}M_1^2M_2^8+\frac{9009}{711025}M_2^{10}\right)\Omega_{12}^{(2)}(2)\\
&\quad-\frac{50053575}{128}\left(\frac{1088}{8975}M_1^6M_2^4+\frac{4752}{8975}M_1^4M_2^6+\frac{2904}{8975}M_1^2M_2^8+\frac{231}{8975}M_2^{10}\right)\Omega_{12}^{(2)}(3)\\
&\quad+\frac{64969619}{384}\left(\frac{16064}{454333}M_1^6M_2^4+\frac{204464}{454333}M_1^4M_2^6+\frac{209352}{454333}M_1^2M_2^8+\frac{24453}{454333}M_2^{10}\right)\Omega_{12}^{(2)}(4)\\
&\quad-\frac{3630289}{96}\left(\frac{6224}{21481}M_1^4M_2^6+\frac{12848}{21481}M_1^2M_2^8+\frac{2409}{21481}M_2^{10}\right)\Omega_{12}^{(2)}(5)\\
&\quad+\frac{953681}{192}\left(\frac{102480}{953681}M_1^4M_2^6+\frac{636272}{953681}M_1^2M_2^8+\frac{214929}{953681}M_2^{10}\right)\Omega_{12}^{(2)}(6)-\frac{29393}{72}\left(\frac{1304}{2261}M_1^2M_2^8+\frac{957}{2261}M_2^{10}\right)\Omega_{12}^{(2)}(7)\\
&\quad+\frac{4699}{216}\left(\frac{1384}{4699}M_1^2M_2^8+\frac{3315}{4699}M_2^{10}\right)\Omega_{12}^{(2)}(8)-\frac{13}{18}\left(M_2^{10}\right)\Omega_{12}^{(2)}(9)+\frac{1}{72}\left(M_2^{10}\right)\Omega_{12}^{(2)}(10)\\
&\quad+\frac{1705275}{16}\left(\frac{368}{11925}M_1^7M_2^3+\frac{4264}{11925}M_1^5M_2^5+\frac{6006}{11925}M_1^3M_2^7+\frac{1287}{11925}M_1M_2^9\right)\Omega_{12}^{(3)}(3)\\
&\quad-\frac{2879591}{36}\left(\frac{328}{1549}M_1^5M_2^5+\frac{924}{1549}M_1^3M_2^7+\frac{297}{1549}M_1M_2^9\right)\Omega_{12}^{(3)}(4)+\frac{289133}{12}\left(\frac{4856}{66723}M_1^5M_2^5+\frac{39988}{66723}M_1^3M_2^7+\frac{21879}{66723}M_1M_2^9\right)\Omega_{12}^{(3)}(5)\\
&\quad-\frac{33982}{9}\left(\frac{614}{1307}M_1^3M_2^7+\frac{693}{1307}M_1M_2^9\right)\Omega_{12}^{(3)}(6)+\frac{3061}{9}\left(\frac{682}{3061}M_1^3M_2^7+\frac{2379}{3061}M_1M_2^9\right)\Omega_{12}^{(3)}(7)\\
&\quad-\frac{52}{3}\left(M_1M_2^9\right)\Omega_{12}^{(3)}(8)+\frac{4}{9}\left(M_1M_2^9\right)\Omega_{12}^{(3)}(9)+\frac{470041}{36}\left(\frac{232}{3287}M_1^6M_2^4+\frac{1768}{3287}M_1^4M_2^6+\frac{1287}{3287}M_1^2M_2^8\right)\Omega_{12}^{(4)}(4)\\
&\quad-\frac{56446}{9}\left(\frac{68}{167}M_1^4M_2^6+\frac{99}{167}M_1^2M_2^8\right)\Omega_{12}^{(4)}(5)+\frac{10682}{9}\left(\frac{1012}{5341}M_1^4M_2^6+\frac{4329}{5341}M_1^2M_2^8\right)\Omega_{12}^{(4)}(6)-104\left(M_1^2M_2^8\right)\Omega_{12}^{(4)}(7)+4\left(M_1^2M_2^8\right)\Omega_{12}^{(4)}(8)\\
&\quad+\frac{5096}{9}\left(\frac{10}{49}M_1^5M_2^5+\frac{39}{49}M_1^3M_2^7\right)\Omega_{12}^{(5)}(5)-\frac{416}{3}\left(M_1^3M_2^7\right)\Omega_{12}^{(5)}(6)+\frac{32}{3}\left(M_1^3M_2^7\right)\Omega_{12}^{(5)}(7)+\frac{16}{3}\left(M_1^4M_2^6\right)\Omega_{12}^{(6)}(6).
\end{aligned}
\tag{146}
$$

Now that the L_1 and L_{12} bracket integrals up to order 5 have been explicitly expressed, they can be used to generate explicit expressions for the simple gas bracket integrals up to order 5 via the rule of Eq. (104) exactly as was done for the general simple gas bracket integral expression of Eqs. (107) and (108). Under the rule of Eq. (104), one has that $M_1 = M_2 = 1/2$ and the simple gas bracket integrals are then mass independent except for the presence of a single m_1 in the simple gas omega integrals. Since, for any given (p, q) all of the terms in the pairs of corresponding L_1 and L_{12} bracket integral expressions have the same total power of the constituent masses, substitution of $M_1 = M_2 = 1/2$ simply yields an additional constant factor for the corresponding pairs of expressions of $(1/2)^{p+q+2}$. The difference in the signs of the terms in the corresponding expressions due to the factor of $(-1)^{\ell}$ has the effect that all terms involving omega integrals with odd values of ℓ cancel exactly and all terms involving omega integrals with even values of ℓ add identically to produce an additional factor of 2 in each surviving term. Thus, it follows that for the simple gas viscosity bracket integrals of Eq. (12) that one has:

$$\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},S^{(0)}_{5/2}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right]_1 = 4\Omega_1^{(2)}(2), \tag{147}$$

$$\left[S^{(0)}_{5/2}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1},S^{(1)}_{5/2}(\mathscr{C}_1^2)\overset{\circ}{\boldsymbol{\mathscr{C}}_1\boldsymbol{\mathscr{C}}_1}\right]_1 = 7\Omega_1^{(2)}(2)-2\Omega_1^{(2)}(3), \tag{148}$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(2)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{63}{8}\Omega_1^{(2)}(2) - \frac{9}{2}\Omega_1^{(2)}(3) + \frac{1}{2}\Omega_1^{(2)}(4), \tag{149}$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(3)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{231}{32}\Omega_1^{(2)}(2) - \frac{99}{16}\Omega_1^{(2)}(3) + \frac{11}{8}\Omega_1^{(2)}(4) - \frac{1}{12}\Omega_1^{(2)}(5), \tag{150}$$

$$\left[S_{5/2}^{(0)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(4)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{3003}{512}\Omega_1^{(2)}(2) - \frac{429}{64}\Omega_1^{(2)}(3) + \frac{143}{64}\Omega_1^{(2)}(4) - \frac{13}{48}\Omega_1^{(2)}(5) + \frac{1}{96}\Omega_1^{(2)}(6), \tag{151}$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{301}{12}\Omega_1^{(2)}(2) - 7\Omega_1^{(2)}(3) + \Omega_1^{(2)}(4), \tag{152}$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(2)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{1365}{32}\Omega_1^{(2)}(2) - \frac{321}{16}\Omega_1^{(2)}(3) + \frac{25}{8}\Omega_1^{(2)}(4) - \frac{1}{4}\Omega_1^{(2)}(5), \tag{153}$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(3)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{6699}{128}\Omega_1^{(2)}(2) - \frac{297}{8}\Omega_1^{(2)}(3) + \frac{385}{48}\Omega_1^{(2)}(4) - \frac{5}{6}\Omega_1^{(2)}(5) + \frac{1}{24}\Omega_1^{(2)}(6), \tag{154}$$

$$\left[S_{5/2}^{(1)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(4)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{109109}{2048}\Omega_1^{(2)}(2) - \frac{52767}{1024}\Omega_1^{(2)}(3) + \frac{11869}{768}\Omega_1^{(2)}(4) - \frac{2405}{1152}\Omega_1^{(2)}(5) + \frac{59}{384}\Omega_1^{(2)}(6) - \frac{1}{192}\Omega_1^{(2)}(7), \tag{155}$$

$$\left[S_{5/2}^{(2)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(2)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{25137}{256}\Omega_1^{(2)}(2) - \frac{1755}{32}\Omega_1^{(2)}(3) + \frac{381}{32}\Omega_1^{(2)}(4) - \frac{9}{8}\Omega_1^{(2)}(5) + \frac{1}{16}\Omega_1^{(2)}(6) + \frac{1}{2}\Omega_1^{(4)}(4), \tag{156}$$

$$\left[S_{5/2}^{(2)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(3)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{155925}{1024}\Omega_1^{(2)}(2) - \frac{56727}{512}\Omega_1^{(2)}(3) + \frac{3795}{128}\Omega_1^{(2)}(4) - \frac{249}{64}\Omega_1^{(2)}(5) + \frac{17}{64}\Omega_1^{(2)}(6) - \frac{1}{96}\Omega_1^{(2)}(7) + \frac{11}{8}\Omega_1^{(4)}(4) - \frac{1}{4}\Omega_1^{(4)}(5), \tag{157}$$

$$\left[S_{5/2}^{(2)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(4)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{3072069}{16384}\Omega_1^{(2)}(2) - \frac{719433}{4096}\Omega_1^{(2)}(3) + \frac{237237}{4096}\Omega_1^{(2)}(4) - \frac{4901}{512}\Omega_1^{(2)}(5) + \frac{891}{1024}\Omega_1^{(2)}(6) - \frac{35}{768}\Omega_1^{(2)}(7) + \frac{1}{768}\Omega_1^{(2)}(8) + \frac{143}{64}\Omega_1^{(4)}(4) - \frac{13}{16}\Omega_1^{(4)}(5) + \frac{1}{16}\Omega_1^{(4)}(6), \tag{158}$$

$$\left[S_{5/2}^{(3)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(3)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{1168167}{4096}\Omega_1^{(2)}(2) - \frac{245025}{1024}\Omega_1^{(2)}(3) + \frac{236005}{3072}\Omega_1^{(2)}(4) - \frac{1573}{128}\Omega_1^{(2)}(5) + \frac{859}{768}\Omega_1^{(2)}(6) - \frac{11}{192}\Omega_1^{(2)}(7) + \frac{1}{576}\Omega_1^{(2)}(8) + \frac{1463}{288}\Omega_1^{(4)}(4) - \frac{11}{8}\Omega_1^{(4)}(5) + \frac{1}{8}\Omega_1^{(4)}(6), \tag{159}$$

$$\left[S_{5/2}^{(3)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(4)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{26951925}{65536}\Omega_1^{(2)}(2) - \frac{13505349}{32768}\Omega_1^{(2)}(3) + \frac{7661797}{49152}\Omega_1^{(2)}(4) - \frac{738569}{24576}\Omega_1^{(2)}(5) + \frac{41005}{12288}\Omega_1^{(2)}(6) - \frac{4175}{18432}\Omega_1^{(2)}(7) + \frac{85}{9216}\Omega_1^{(2)}(8) - \frac{1}{4608}\Omega_1^{(2)}(9) + \frac{23881}{2304}\Omega_1^{(4)}(4) - \frac{4745}{1152}\Omega_1^{(4)}(5) + \frac{37}{64}\Omega_1^{(4)}(6) - \frac{1}{32}\Omega_1^{(4)}(7), \tag{160}$$

$$\left[S_{5/2}^{(4)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1, S_{5/2}^{(4)}(\mathscr{C}_1^2)\boldsymbol{\mathcal{C}}_1\mathring{\boldsymbol{\mathcal{C}}}_1\right]_1 = \frac{711736025}{1048576}\Omega_1^{(2)}(2) - \frac{50053575}{65536}\Omega_1^{(2)}(3) + \frac{64969619}{196608}\Omega_1^{(2)}(4) - \frac{3630289}{49152}\Omega_1^{(2)}(5) + \frac{953681}{98304}\Omega_1^{(2)}(6) - \frac{29393}{36864}\Omega_1^{(2)}(7) + \frac{4699}{110592}\Omega_1^{(2)}(8) - \frac{13}{9216}\Omega_1^{(2)}(9) + \frac{1}{36864}\Omega_1^{(2)}(10) + \frac{470041}{18432}\Omega_1^{(4)}(4) - \frac{28223}{2304}\Omega_1^{(4)}(5) + \frac{5341}{2304}\Omega_1^{(4)}(6) - \frac{13}{64}\Omega_1^{(4)}(7) + \frac{1}{128}\Omega_1^{(4)}(8) + \frac{1}{96}\Omega_1^{(6)}(6). \tag{161}$$

9. Discussion and conclusions

Our purpose in this series of papers [17–21] has been to explore the use of Sonine polynomial expansions to obtain high-precision numerical results for the transport coefficients and the related Chapman–Enskog functions for simple gases and gas mixtures that are free of numerical error. In the current work we have presented summational expressions for the viscosity-related bracket integrals needed to compute the associated Chapman–Enskog solutions to any arbitrary order of expansion in terms of Sonine polynomials. The summational nature of the expressions derived in this work will yield greatly improved computational efficiencies in determinations of the viscosity-related matrix elements associated with our current *Mathematica*®-based programs and will undoubtedly prove substantially more conducive to use in computations that do not involve the use of *Mathematica*® but which, rather, employ the Fortran, C++, or other programming environments directly. In addition, the derived expressions which we have reported here clearly show exactly which omega integrals are needed in each (p, q) element of the coefficient matrix for a given order of the Sonine polynomial expansion approximation, m, which we have found requires some additional effort to extract using the term-by-term, expand-and-compare method that we have used in our previous work [17–20] based on the methodology originally suggested by Chapman and Cowling [1]. The work completed so far enables future investigations on ternary and more complicated gas mixtures and use of realistic potentials, and we plan to report on such results in the future.

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